Wolf Theorem 8.6 — the Doubly-Substochastic Half of Birkhoff

2026-08-09

Verdict: scope-restriction (open).

1 The source statement

A nonnegative $d \times d$ matrix is doubly stochastic when every row sum and every column sum equals 1. It is doubly substochastic when every row sum and every column sum is at most 1. A permutation matrix has exactly one entry equal to 1 in each row and column. A partial permutation matrix is a $\{0, 1\}$-matrix with at most one entry equal to 1 in each row and column.

Wolf's Theorem 8.6, called "Birkhoff's theorem", states two characterizations in one theorem environment [Wol12, Chapter 8, Theorem 8.6]. The local source is `Notes/WolfNoteTexSource/ch08_distance_measures.tex`, lines 263–266.

DS The set of $d \times d$ doubly stochastic matrices is the convex hull of the $d \times d$ permutation matrices, and its extreme points are exactly the permutation matrices.

Sub The set of $d \times d$ doubly substochastic matrices is the convex hull of the $d \times d$ partial permutation matrices, and its extreme points are exactly the partial permutation matrices.

2 Formalized component

The doubly-stochastic part **DS** is formalized by two declarations:

- `TNLean.wolf_birkhoff_doublyStochastic_convexHull` states that the doubly stochastic matrices form the convex hull of the permutation matrices;
- `TNLean.wolf_birkhoff_extremePoints_doublyStochastic` states that the extreme points of the doubly stochastic matrices are exactly the permutation matrices.

Both declarations are in `QICLean/Analysis/Birkhoff`. They directly reuse Mathlib's `doublyStochastic_eq_convexHull_permMatrix` and `extremePoints_doublyStochastic`.

3 Missing component

The doubly-substochastic part **Sub** of Theorem 8.6 [Wol12] is not formalized. A complete treatment requires the following definitions and results.

1. Define the predicate "doubly substochastic" for a nonnegative square matrix whose row and column sums are bounded by 1.
2. Define partial permutation matrices as square $\{0, 1\}$-matrices with at most one 1 in each row and each column.
3. Prove that the doubly substochastic matrices form the convex hull of the partial permutation matrices and that their extreme points are exactly the partial permutation matrices.

The substochastic polytope contains the stochastic polytope as a face, but its support may omit rows or columns. Its extreme-point proof must therefore replace permutation matrices by partial permutation matrices.

4 Elimination plan

First, adapt Mathlib's `doublyStochastic` predicate to a predicate `doublySubstochastic`, replacing the row-sum and column-sum equalities by upper bounds. Then define partial permutation matrices over $\mathrm{Fin}\, d$ and prove that they are doubly substochastic.

For the convex-hull statement, let M be a $d \times d$ doubly substochastic matrix and define

$$u_i = \sum_j M_{ij}, \qquad v_j = \sum_i M_{ij}, \tag{1}$$

$$r_i = 1 - u_i, \qquad c_j = 1 - v_j, \tag{2}$$

$$s = \sum_{i,j} M_{ij}. \tag{3}$$

If $s > 0$, define S by

$$S_{ji} = \frac{v_j u_i}{s}. \tag{4}$$

If $s = 0$, set

$$S = 0. \tag{5}$$

For $s > 0$, the row and column sums of S are

$$\sum_i S_{ji} = v_j, \tag{6}$$

$$\sum_j S_{ji} = u_i. \tag{7}$$

The same identities hold when $s = 0$, because then $M = 0$ and all u_i and v_j vanish.

Define the block matrix

$$\widetilde{M} = \begin{pmatrix} M & \operatorname{diag}(r) \\ \operatorname{diag}(c) & S \end{pmatrix}. \tag{8}$$

Its row and column sums satisfy

$$\sum_b \widetilde{M}_{ab} = 1, \tag{9}$$

$$\sum_a \widetilde{M}_{ab} = 1. \tag{10}$$

Thus $\widetilde{M}$ is a $2d \times 2d$ doubly stochastic matrix. Applying the formalized doubly-stochastic Birkhoff theorem to $\widetilde{M}$ and restricting each permutation matrix to the original $d \times d$ block yields a convex decomposition of M into partial permutation matrices.

The final step is to adapt the stochastic extreme-point argument and prove that the extreme points of the doubly substochastic set are precisely the partial permutation matrices.

5 Conclusion

This note records a scope restriction. The doubly-stochastic part of Wolf's Theorem 8.6 [Wol12], including both the convex-hull and extreme-point characterizations, is fully formalized. The definition and polytope theory of doubly substochastic matrices, including the partial-permutation extreme-point characterization, remain unformalized. Tracking: TNLean issue #3959.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.