The Spectral-Radius Identification in Wolf Theorem 6.5

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Verdict: open-gap (open).

1 Source statement

For a linear map $T : M_d(\mathbb{C}) \to M_d(\mathbb{C})$, write $\varrho(T)$ for its spectral radius. For $X \in M_d(\mathbb{C})$, write $X \succeq 0$ when X is positive semidefinite and $X \succ 0$ when it is positive definite. Wolf's Theorem 6.5 states that every positive linear map $T : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ admits a nonzero positive semidefinite eigenvector at its spectral radius:

$$T(X) = \varrho(T)X, \tag{1}$$
$$X \succeq 0, \qquad X \neq 0. \tag{2}$$

This is Theorem 6.5 of Ref. [Wol12]. The local source is `Notes/WolfNoteTexSource/ch06_spectral_properties.tex`, lines 743–747.

2 What is formalized

Two formal results establish separate parts of the source assertion. First, every positive linear map $E : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ has a nonnegative eigenvalue with a nonzero positive semidefinite eigenvector:

$$E(\rho) = r\rho, \tag{3}$$
$$\rho \succeq 0, \qquad \rho \neq 0, \qquad r \geq 0. \tag{4}$$

This is formalized by `exists_posSemidef_eigenvector_general`.[1]

Second, if a positive map has a positive eigenvalue with a positive definite eigenvector, then that eigenvalue equals the spectral radius:

$$E(\rho) = r\rho, \quad \rho \succ 0, \quad r > 0 \quad \Longrightarrow \quad \varrho(E) = r. \tag{5}$$

[1] Formal module: `QICLean/Channel/PerronFrobenius/Existence`. The stronger declaration `exists_posSemidef_eigenvector` assumes that E annihilates no nonzero positive semidefinite matrix and returns $r > 0$.

This result requires neither irreducibility nor complete positivity. It is formalized by `spectralRadius_eq_of_posDef_eigenvector_of_positive`.[2]

The declaration `exists_wolfTheorem63_of_irreducible_positive` constructs a positive definite Perron eigenvector and identifies its eigenvalue with the spectral radius for every irreducible positive map. The source-shaped variant `exists_wolfTheorem63_of_irreducible_positive_of_ne_zero` assumes the map is nonzero and returns an eigenvalue $r > 0$.

3 Comparison

The existence result (3) does not identify r with the spectral radius. Such an identification is false for an arbitrary nonnegative eigenvalue of a positive map.

For example, define the completely positive map $E : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$E(X) = X_{11}\mathbb{1}. \tag{6}$$

It satisfies

$$E(\mathbb{1}) = \mathbb{1}, \tag{7}$$

$$\varrho(E) = 1. \tag{8}$$

However, the nonzero positive semidefinite matrix

$$X = |2\rangle\langle 2| \tag{9}$$

lies in the kernel:

$$E(X) = 0, \tag{10}$$

$$0 < \varrho(E). \tag{11}$$

Thus a general existence theorem may return $r = 0$ even when the spectral radius is positive.

The identification theorem (5) requires a positive definite eigenvector. Wolf's Theorem 6.5 [Wol12] must also cover reducible maps, for which a spectral-radius eigenvector may be singular. The irreducible case is complete; the unresolved step is the extension to arbitrary reducible positive maps.

4 Routes to a complete proof

A proof for general positive maps can proceed by either of the following routes.

1. Use the resolvent. For every $\lambda > \varrho(T)$, prove that $(\lambda - T)^{-1}$ preserves positivity. Then extract a positive semidefinite eigenvector from the pole at $\lambda = \varrho(T)$. This route requires holomorphic functional calculus and spectral projections.

[2] Formal module: `QICLean/Channel/Irreducible/SpectralRadius`.

2. Use invariant faces of the positive cone. Decompose the positive map into irreducible components, apply the Perron–Frobenius theorem to each component, and select a component with maximal spectral radius. This route requires an invariant-face decomposition for positive maps.

Wolf's Theorem 6.3 for irreducible positive maps [Wol12] is formalized. Its proof follows the lower-maximizer argument in Equations (6.31)–(6.33) of Ref. [Wol12] and the positive-unital similarity argument; it does not require a separate pointwise Collatz–Wielandt theorem. The remaining work is exactly the reducible-map step described above.

5 Conclusion

Wolf's Theorem 6.5 [Wol12] remains an open gap. A nonnegative eigenvalue with a positive semidefinite eigenvector exists for every positive map, and the spectral-radius identification is established for irreducible positive maps. The unresolved case is a reducible positive map whose spectral-radius eigenvector may be singular.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.