Implicit Nilpotent-Space Classification in the Low-Dimensional Quantum Wielandt Corollary

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Verdict: open-gap (wip).

1 The assertion

For a Kraus family on $\mathbb{C}^d$, let $\mathcal{K}_1 \subseteq M_d(\mathbb{C})$ denote its one-step linear span. The corollary following Wolf's quantum Wielandt inequality states that, when $d = 2$ or $d = 3$, every primitive quantum channel has an element of $\mathcal{K}_1$ with a nonzero eigenvalue. It then concludes that the channel's Wielandt index is at most d^2 [3, Chapter 6, lines 1110–1126]. The same low-dimensional observation appears after Theorem 1 of Ref. [2].

The proof in Ref. [3] invokes a classification of nilpotent matrix spaces but does not prove that classification or resolve the bibliography placeholders in the local source. In dimension two, the required claim is

$$\mathcal{N} \subseteq M_2(\mathbb{C}) \text{ linear and nilpotent} \quad \Longrightarrow \quad \dim \mathcal{N} \leq 1. \tag{1}$$

In dimension three, Wolf describes two similarity types for nilpotent spaces of dimension greater than one: the strictly upper-triangular space and an exceptional two-dimensional space [3, Chapter 6]. The strictly upper-triangular type is incompatible with trace preservation, while the exceptional type yields a channel with degenerate peripheral spectral radius [3, Chapter 6]. The lecture notes do not contain the classification or the two obstruction arguments.

Fasoli's classification of nilpotent linear spaces is a relevant external source [1]. The exact chain of results required for Wolf's low-dimensional corollary must nevertheless be reconstructed and checked.

2 Formalized part

The dimension-two classification is complete. `QICLean.finrank_le_one_of_isNilpotent_submodule_fin_two` proves (1), and `QICLean.exists_common_ker_of_isNilpotent_submodule_fin_two` proves the stronger common-kernel statement used for channels. Combined with trace preservation, `Kraus.exists_nonzero_eigenvector_mem_wordSpan_one_fin_two` proves that the one-step span of every trace-preserving Kraus family on $M_2(\mathbb{C})$ contains an element with a nonzero eigenvalue.

For $d = 3$, `QICLean.exists_common_ker_of_forall_sq_eq_zero` proves the square-zero branch, and `Kraus.exists_mem_wordSpan_one_sq_ne_zero_fin_three` excludes that branch for a trace-preserving family. The remaining nilpotency-index-three branch is not formalized. It includes both the exceptional similarity type and the proof that this type obstructs primitivity.

3 Remaining gap

A source-faithful proof still requires the following ingredients.

1. Prove the remaining 3×3 classification: a nilpotent linear space containing an element of nilpotency index three either has a common kernel or is similar to the exceptional two-dimensional space. Then prove that the exceptional Kraus pattern is incompatible with Wolf's primitivity condition.

2. Prove in QICLean the conditional implication from a nonzero eigenvalue in the one-step span to full word span at length d^2. The current QICLean checkout contains the eigenvector-spreading component but not the assembled bound. The numbering-coverage document records Wolf's Theorem 6.9 [3, Chapter 6] as belonging to the downstream tensor-network layer.

Until both ingredients are present, the source corollary must not carry a `\leanok` marker in the QICLean blueprint. This work is tracked by GitHub issue #65.

4 Conclusion

The $d = 2$ nilpotent-space argument and the square-zero branch for $d = 3$ are formalized. The nilpotency-index-three classification, its exceptional primitivity obstruction, and the assembled d^2 word-span bound remain open. Consequently, the low-dimensional corollary of Ref. [3] remains an open gap while elimination is in progress.

References

[1] Maria A. Fasoli. Classification of nilpotent linear spaces in $m(4;\mathbb{C})$. *Communications in Algebra*, 25(6):1919–1932, 1997.

[2] M. Sanz, D. Pérez-García, M. M. Wolf, and J. I. Cirac. A quantum version of wielandt's inequality. *IEEE Transactions on Information Theory*, 56(9):4668–4673, 2010.

[3] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.