The Phase Index in the Weighted Cesàro Formula, Wolf Equation (6.15)

2026-08-16

Verdict: local-correction (resolved).

1 The source sentence

Let $T : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ be a trace-preserving positive linear map. After expressing the asymptotic map T_∞ as a Cesàro mean in Equation (6.14), Wolf writes

$$T_\phi = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{k:|\lambda_k|=1} (\overline{\lambda_k} T)^n. \tag{1}$$

This is Equation (6.15) of Ref. [Wol12]. The local source is `Notes/WolfNoteTexSource/ch06_spectral_properties.tex`, lines 258–264.

In this chapter, k indexes the Jordan blocks of the superoperator $\widehat{T}$. Equations (6.4)–(6.5) of Ref. [Wol12], at lines 111–139 of the local source, use

$$\widehat{T} = X \left(\bigoplus_{k=1}^{K} J_k(\lambda_k) \right) X^{-1}. \tag{2}$$

The number of blocks carrying an eigenvalue λ is its geometric multiplicity, so distinct block indices may carry the same eigenvalue [Wol12, Chapter 6].

2 Correction forced by multiplicities

The summand $(\overline{\lambda_k} T)^n$ depends only on λ_k, not on the block indexed by k. Consequently, the literal block-indexed sum in (1) counts each peripheral eigenvalue with its geometric multiplicity.

For a counterexample, take the identity channel $T = \mathbb{1}$ on $M_D(\mathbb{C})$ with $D > 1$. Every matrix is an eigenvector with eigenvalue 1, and the Jordan decomposition of $\widehat{T}$ has D^2 one-dimensional blocks carrying that eigenvalue.

The literal right-hand side becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} D^2 \mathbb{1} = D^2 \mathbb{1}, \tag{3}$$

$$D^2 \mathbb{1} \neq \mathbb{1} = T_\phi. \tag{4}$$

Thus Equation (6.15) of Ref. [Wol12] is false when k is read literally as a Jordan-block index.

Let $\mathcal{P}(T)$ denote the set of distinct eigenvalues of T on the unit circle. The corrected formula is

$$T_\phi = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{\lambda \in \mathcal{P}(T)} (\overline{\lambda} T)^n. \tag{5}$$

Each peripheral phase is counted once.

This reading agrees with the surrounding spectral decomposition. Peripheral Jordan blocks of a trace-preserving positive map are one-dimensional [Wol12, Chapter 6]. Hence the peripheral subspace is the direct sum of the eigenspaces associated with the distinct peripheral eigenvalues. If Y belongs to the eigenspace of λ, then the summand indexed by $\mu \in \mathcal{P}(T)$ satisfies

$$(\overline{\mu} T)^n (Y) = (\overline{\mu}\lambda)^n Y. \tag{6}$$

Its Cesàro mean obeys

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (\overline{\mu}\lambda)^n = \begin{cases} 1, & \mu = \lambda, \\ 0, & \mu \neq \lambda. \end{cases} \tag{7}$$

Therefore one summand for each distinct peripheral eigenvalue recovers the peripheral projection.

The neighboring expressions for $\widehat{T}_\infty$, $\widehat{T}_\phi$, and $\widehat{T}_\varphi$ in Equations (6.11)–(6.13) of Ref. [Wol12] are also indexed by Jordan blocks, but their summands contain the block projections P_k. Those formulas retain block dependence and are correct as printed. Only Equation (6.15), where the summand no longer depends on the block, requires a deduplicated index.

3 Formalization

The formal development averages over the set of eigenvalues of modulus one, so each distinct phase occurs once.[1] This correction changes no hypothesis or intended conclusion of Equation (6.15) of Ref. [Wol12]; it changes only the multiplicity convention for the inner index.

[1] Formal declarations: `Module.End.weightedCesaroMean` for the average, and `IsPositiveMap.tendsto_weightedCesaroMean_peripheralProjection`, `IsPositiveMap.tendsto_endEquiv_weightedCesaroMean_peripheralProjection`, `IsPositiveMap.tendsto_weightedCesaroMean_toFinset_peripheralProjection`, and `IsPositiveMap.tendsto_endEquiv_weightedCesaroMean_toFinset_peripheralProjection` in `QICLean/Channel/Peripheral/WeightedCesaro`.

4 Conclusion

Equation (6.15) of Ref. [Wol12] requires a local correction: the inner sum must range over distinct peripheral eigenvalues, not peripheral Jordan blocks. The identity channel gives the contradiction (4) under the literal block-indexed reading. The deduplicated formula (5) has the source's intended hypotheses and conclusion.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.