The Missing Order Hypothesis in Wolf's Unitary Comparison

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Verdict: false-source (resolved).

1 The assertion

Let $A, B \in M_d(\mathbb{C})$ be Hermitian, and write $\lambda_1^{\downarrow}(A), \ldots, \lambda_d^{\downarrow}(A)$ for the eigenvalues of A in decreasing order. The paragraph preceding Wolf's Proposition 5.3 assumes $A \leq B$ and applies Weyl's monotonicity theorem in the form

$$\lambda_j^{\downarrow}(B) \geq \lambda_j^{\downarrow}(A) + \lambda_d^{\downarrow}(B - A), \qquad 1 \leq j \leq d. \tag{1}$$

This is Equation (5.56) of Ref. [Wol12]. Since $B - A \geq 0$, its smallest eigenvalue is nonnegative. Hence

$$\lambda_j^{\downarrow}(A) \leq \lambda_j^{\downarrow}(B), \qquad 1 \leq j \leq d. \tag{2}$$

Proposition 5.3 then drops the hypothesis $A \leq B$. It claims that arbitrary Hermitian $A, B \in M_d(\mathbb{C})$ admit a unitary U, independent of f, such that

$$f(A) \leq U f(B) U^{\dagger} \tag{3}$$

for every non-decreasing function $f : I \to \mathbb{R}$, where I is an interval containing both spectra. This is Equation (5.57) of Ref. [Wol12].[1]

2 The point at issue

The printed proposition is false. In dimension one, set

$$A = [1], \qquad B = [0], \qquad I = [0, 1], \qquad f(x) = x. \tag{4}$$

[1]Source: [Wol12, Proposition 5.3 and Equations (5.56)–(5.57)]; local source `Notes/WolfNoteTexSource/ch05_schwarz_inequalities.tex`, lines 1119–1135. Tracked in GitHub issue #27.

Every one-dimensional unitary is a scalar phase, so

$$Uf(B)U^\dagger = 0, \tag{5}$$

$$f(A) \le Uf(B)U^\dagger \iff 1 \le 0. \tag{6}$$

Thus the hypotheses printed in Proposition 5.3 of Ref. [Wol12] do not imply its conclusion.

The source proof also requires the omitted order assumption. Equation (5.56) of Ref. [Wol12] assumes $A \le B$, and this assumption yields the ordered eigenvalue comparison (2). Without it, the proof supplies no comparison between the two ordered spectra.

3 The corrected statement and source proof

The source argument proves the following corrected proposition:

$$\begin{aligned} A \le B \quad \Longrightarrow \quad \exists U \in M_d(\mathbb{C})\ \Big[& U^\dagger U = UU^\dagger = \mathbb{1} \qquad (7)\\ & \wedge \forall I \subseteq \mathbb{R}\ \forall f : I \to \mathbb{R},\\ & \big(I \text{ is an interval } \wedge \ \mathrm{spec}(A) \cup \mathrm{spec}(B) \subseteq I \ \wedge \ f \text{ is non-decreasing on } I\big)\\ & \qquad \Longrightarrow f(A) \le Uf(B)U^\dagger \Big]. \qquad (8) \end{aligned}$$

The unitary is chosen before the interval and the function, so the same U works for every admissible I and f.

Choose unitaries V_A and V_B whose columns are eigenvectors arranged in decreasing eigenvalue order. Then

$$A = V_A \operatorname{diag}\big(\lambda_1^\downarrow(A), \dots, \lambda_d^\downarrow(A)\big) V_A^\dagger, \tag{9}$$

$$B = V_B \operatorname{diag}\big(\lambda_1^\downarrow(B), \dots, \lambda_d^\downarrow(B)\big) V_B^\dagger. \tag{10}$$

The order hypothesis and Equation (5.56) of Ref. [Wol12] give (2). Monotonicity of f therefore implies

$$f(\lambda_j^\downarrow(A)) \le f(\lambda_j^\downarrow(B)), \qquad 1 \le j \le d. \tag{11}$$

Set

$$U = V_A V_B^\dagger. \tag{12}$$

Functional calculus in the two eigenbases gives

$$f(A) = V_A \operatorname{diag}\big(f(\lambda_1^\downarrow(A)), \dots, f(\lambda_d^\downarrow(A))\big) V_A^\dagger, \tag{13}$$

$$Uf(B)U^\dagger = V_A \operatorname{diag}\big(f(\lambda_1^\downarrow(B)), \dots, f(\lambda_d^\downarrow(B))\big) V_A^\dagger. \tag{14}$$

The diagonal comparison (11), transported by the unitary congruence $X \mapsto V_A X V_A^\dagger$, proves (8).

4 Formalization

The cross-matrix Weyl comparison is formalized by `LinearMap.IsSymmetric.eigenvalues_le_of_sub_isPositive`, with matrix forms `Matrix.IsHermitian.eigenvalues_mono` and `Matrix.IsHermitian.eigenvalues_mono`.[2] Its finite-dimensional variational proof intersects the span of the first $j+1$ eigenvectors of A with the span of the last $d-j$ eigenvectors of B. Their dimensions sum to $d+1$, so they contain a common nonzero vector x. The ordered spectral expansions and positivity of $B-A$ yield

$$\lambda_j^{\downarrow}(A)\|x\|^2 \le \mathrm{Re}\langle Ax|x\rangle, \tag{15}$$

$$\mathrm{Re}\langle Ax|x\rangle \le \mathrm{Re}\langle Bx|x\rangle, \tag{16}$$

$$\mathrm{Re}\langle Bx|x\rangle \le \lambda_j^{\downarrow}(B)\|x\|^2. \tag{17}$$

Since $x \neq 0$, these inequalities prove the required cross-matrix eigenvalue comparison.

The corrected proposition is formalized by `Matrix.IsHermitian.exists_unitary_cfc_le_cfc_of_le`.[3] The proof chooses the unitary from the two decreasingly ordered eigenbases and uses the formalized Weyl comparison for every interval and every non-decreasing function.

5 Conclusion

Wolf's Proposition 5.3 is false as printed because it omits $A \le B$. The scalar example (4) satisfies the printed hypotheses and contradicts the conclusion. Adding $A \le B$ makes the proposition follow from Equation (5.56) of Ref. [Wol12]: compare the ordered spectra, apply scalar monotonicity of f, and align the eigenbases with one unitary. Both the cross-matrix Weyl comparison and this corrected unitary comparison are formalized without assuming a pointwise eigenvalue comparison.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.

[2] Formal module: `QICLean/Analysis/WeylMonotonicity`.

[3] Formal module: `QICLean/Channel/Schwarz/UnitaryComparison`.