The Inverse on the Range in the Two-Variable Operator Schwarz Inequality

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Verdict: open-gap (resolved).

1 Source statement

Let

$$T\colon M_d(\mathbb{C}) \to M_{d'}(\mathbb{C}), \qquad E = T^*\colon M_{d'}(\mathbb{C}) \to M_d(\mathbb{C}), \tag{1}$$

where T is two-positive and E is its trace adjoint. Wolf's Theorem 5.3 asserts that, for every $A, B \in M_{d'}(\mathbb{C})$,

$$E(A^\dagger B)E(B^\dagger B)^{-1}E(B^\dagger A) \leq E(A^\dagger A), \tag{2}$$

where the inverse is taken on the range of $E(B^\dagger B)$ [Wol12, Chapter 5, Theorem 5.3].[1]

Rectangular two-positivity is invariant under the trace adjoint, so $E = T^*$ is also two-positive. The formal theorem retains the independent input and output dimensions used by Wolf and assumes no additional range witness. It represents the inverse on the range by the Moore–Penrose inverse.

2 The range inverse

Define

$$P = E(A^\dagger A), \qquad Z = E(A^\dagger B), \qquad R = E(B^\dagger B). \tag{3}$$

Two-positivity of E gives the block inequality

$$\begin{pmatrix} P & Z \\ Z^\dagger & R \end{pmatrix} \geq 0. \tag{4}$$

[1] Local source: `Notes/WolfNoteTexSource/ch05_schwarz_inequalities.tex`, lines 180–190.

The kernel clause of the Schur-complement theorem applied to Eq. 4 gives

$$\ker R \subseteq \ker Z. \tag{5}$$

Let R^+ denote the Moore–Penrose inverse of R, and define

$$s_R = RR^+ \tag{6}$$

$$= R^+R. \tag{7}$$

Because $R \geq 0$, s_R is the orthogonal projection onto $\operatorname{ran} R = (\ker R)^\perp$. The kernel inclusion in Eq. 5 is equivalent to the support-absorption identities

$$Zs_R = Z, \tag{8}$$

$$s_R Z^\dagger = Z^\dagger. \tag{9}$$

Consequently,

$$R(R^+Z^\dagger) = (RR^+)Z^\dagger \tag{10}$$

$$= s_R Z^\dagger \tag{11}$$

$$= Z^\dagger. \tag{12}$$

For every vector v, the vector $R^+Z^\dagger v$ is therefore a preimage of $Z^\dagger v$ under R. Equation 12 gives exactly the inverse-on-the-range operation required in Eq. 2. No separate range–kernel duality argument is needed.

The formal proof establishes Eq. 5, then Eqs. 8 and 9, and finally Eq. 12.

3 Derivation of the inequality

Fix a vector v and define

$$w = R^+Z^\dagger v. \tag{13}$$

Equation 12 gives

$$Rw = Z^\dagger v. \tag{14}$$

The pseudoinverse-free quadratic-form consequence of Eq. 4 then yields

$$v^\dagger P v \geq v^\dagger Z w \tag{15}$$

$$= v^\dagger Z R^+ Z^\dagger v. \tag{16}$$

Because this inequality holds for every v,

$$ZR^+Z^\dagger \leq P. \tag{17}$$

Substituting the definitions from Eq. 3 into Eq. 17 gives Eq. 2, with R^+ realizing the inverse on the range.[2]

[2] Formal declaration: `schwarz_two_variable_unconditional_of_traceAdjointMap` in the namespace `SchwarzTwoVariable`.

4 Verdict

No hypothesis or dimensional restriction remains relative to Wolf's Theorem 5.3 [Wol12, Chapter 5, Theorem 5.3]. The formal theorem records the positive map as the trace adjoint $E = T^*$ of a two-positive reverse-direction map T. Rectangular two-positivity of E follows from invariance under the trace adjoint. The Moore–Penrose inverse realizes Wolf's "inverse on the range", and the required range inclusion follows from support-projection absorption. The earlier proposed Euclidean range–kernel-duality argument is unnecessary. The former gap is therefore resolved.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.