Wolf Chapter 5 Special-Function Jensen and Lieb Concavity

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Verdict: false-source (resolved).

1 Scope

This note records two resolved parts of Chapter 5 of Ref. [Wol12]. First, the formal development proves the special-function Jensen inequalities corresponding to Wolf's Corollary 5.2, after correcting its false logarithmic clause. Second, it proves the finite-dimensional Ando–Lieb theorem throughout the exponent range stated in Wolf's Theorem 5.15. The separate scope gap for Wolf's arbitrary operator-convex and operator-monotone functions is recorded in `docs/paper-gaps/wolf_ch5_operator_convexity_scope.tex`.

All operators act on a finite-dimensional Hilbert space, identified with $D \times D$ complex matrices. The relation $A \leq B$ is the Loewner order. A linear map T is positive if it preserves positive semidefiniteness, subunital if $T(\mathbb{1}) \leq \mathbb{1}$, and unital if $T(\mathbb{1}) = \mathbb{1}$. For $A \geq 0$, the power A^p is defined by continuous functional calculus; for $A > 0$, the same applies to $\log A$.

The relevant formal declarations are proved theorems rather than axioms.[1]

2 Positive-map Jensen inequalities

2.1 The assertions

For a positive subunital map T and a positive semidefinite matrix A, the development proves

$$T(A^p) \leq (T(A))^p, \qquad 0 \leq p \leq 1, \tag{1}$$
$$(T(A))^p \leq T(A^p), \qquad 1 \leq p \leq 2. \tag{2}$$

For a positive unital map T and $A > 0$, it proves

$$T(\log A) \leq \log(T(A)). \tag{3}$$

[1] Formal declarations: `posMap_rpow_concave_jensen`, `posMap_rpow_convex_jensen`, `posMap_log_concave_jensen`, and `lieb_concavity_posDef` in `QICLean/Analysis/LiebConcavity`.

These are the operator-valued Jensen inequalities associated with Wolf's Corollary 5.2 [Wol12, Corollary 5.2].[2]

Equation 1 expresses operator concavity of $x \mapsto x^p$ for $p \in [0, 1]$. Equation 2 expresses operator convexity for $p \in [1, 2]$. Equation 3 expresses operator concavity of the logarithm. The logarithmic theorem requires unitality; Wolf's common subunital hypothesis is insufficient.

2.2 The special-function proof route

The formal library supplies integral representations for real powers, including `CFC.exists_measure_nnrpow_eq_integral_cfc_rpowIntegrand` and its $[1, 2]$ companion in `Mathlib/Analysis/SpecialFunctions/ContinuousFunctionalCalculus/Rpow/IntegralRepresentation.lean`. It also supplies operator concavity of $x \mapsto x^p$ on $[0, 1]$ through `CFC.concaveOn_rpow`, operator concavity of the logarithm through `CFC.concaveOn_log`, and the required Bochner-integral convexity transfer for C^*-algebras. Operator convexity of $x \mapsto x^p$ on $[1, 2]$ is recorded as `CFC.convexOn_rpow` in `QICLean/Analysis/RpowConvexity`.

A general passage from operator concavity of f to

$$T(f(A)) \leq f(T(A)) \tag{4}$$

for positive maps is the positive-map operator Jensen principle used in Wolf's Theorems 5.10–5.13 [Wol12, Theorems 5.10–5.13]. Mathlib 4.31 contains no general theorem of this scope for positive maps. Instead, the concave real-power case is proved by the Löwner-integrand theorem `TNLean.OperatorJensen.positiveMap_rpowIntegrand_jensen` and ordered Bochner integration.

The logarithmic theorem follows by taking the right limit $p \to 0^+$ in the concave real-power inequality and using `CFC.tendsto_cfc_rpow_sub_one_log`. The convex real-power case uses the same positive-map Jensen method with the convex integrand for $p \in [1, 2]$.

2.3 Bare positivity is sufficient

The source statements assume that T is positive, not completely positive [Wol12, Corollary 5.2 and Theorems 5.10–5.13]. A proof that required complete positivity would therefore strengthen the source hypotheses.

Wolf avoids this strengthening by restricting T to the abelian C^*-algebra generated by A and $\mathbb{1}$. On this algebra every positive map is automatically completely positive, and the restricted map has a completely positive extension to the full matrix algebra [Wol12, Propositions 1.6–1.7 and Theorems 5.10–5.13]. Thus the source proof reduces the bare positive-map case to a completely positive map without changing the theorem's hypotheses.

[2] Map-facing interfaces: `IsPositiveMap.rpow_concave_jensen`, `IsPositiveMap.rpow_convex_jensen`, and `IsPositiveMap.log_concave_jensen` in `QICLean/Channel/Schwarz/OperatorConvexity`.

The formal finite-POVM scaffold implements the corresponding matrix reduction.[3] Let $P_1, \ldots, P_n$ be the spectral projections of A. Positivity of T gives positive matrices $T(P_i)$, and linearity gives

$$\sum_{i=1}^{n} T(P_i) = T\left(\sum_{i=1}^{n} P_i\right) \tag{5}$$
$$= T(\mathbb{1}). \tag{6}$$

This is the identity used in Wolf's block-positivity argument [Wol12, Theorem 5.12]. The construction uses a positive family and a defect block, not a global Kraus or Stinespring representation of T. It therefore applies under `IsPositiveMap` and subunitality.

The pointwise finite-POVM resolvent inequality `povm_resolvent_inv_le` is integrated through the real-power representation to prove `posMap_rpow_concave_jensen`. Almost-everywhere Loewner inequalities are integrated using `integral_mono_ae`. The positive-cone specialization is `TNLean.OperatorJensen.integral_nonneg_matrix_of_ae`, which states that the Bochner integral of an almost-everywhere positive semidefinite matrix-valued function is positive semidefinite.

The logarithmic theorem `posMap_log_concave_jensen` then follows by applying `CFC.tendsto_cfc_rpow_sub_one_log` to both A and $T(A)$ and using closedness of the finite-dimensional Loewner cone. The convex theorem `posMap_rpow_convex_jensen` uses the corresponding convex integrand and the input `CFC.convexOn_rpow`.

2.4 The false logarithmic clause

Wolf's Corollary 5.2 prints the logarithmic inequality under the same subunital hypothesis as the real-power inequalities:

$$T(\log A) \leq \log(T(A)), \qquad\qquad A > 0. \tag{7}$$

This is item 3 of Corollary 5.2 in Ref. [Wol12]. It is false.

In dimension one, choose $0 < c < 1$, define $T_c(x) = cx$, and set $A = 1$. Then

$$T_c(1) = c \leq 1, \tag{8}$$
$$T_c(\log 1) = 0, \tag{9}$$
$$\log(T_c(1)) = \log c < 0. \tag{10}$$

Consequently, Eq. 7 would assert $0 \leq \log c$, a contradiction. The corrected declaration `IsPositiveMap.log_concave_jensen` assumes

$$T(\mathbb{1}) = \mathbb{1} \tag{11}$$

[3] Formal development: `QICLean/Channel/Schwarz/OperatorJensenAux`, containing finite-POVM compression, the Schur-complement inverse bound, and the resolvent inequality.

and proves Eq. 3. This is a source correction rather than a formalization-only restriction.

The real-power theorems retain Wolf's bare subunital hypothesis. Former pass-through declarations that merely repackaged them in the exact exponent parameterization of items 1–2 were removed as unused.

2.5 Jensen verdict

Both real-power Jensen inequalities are proved under the source hypotheses. The logarithmic inequality is proved under the necessary unital correction. The concave power proof uses finite-POVM compression, the pointwise resolvent inequality, and ordered Bochner integration. The logarithmic proof takes the right limit of the concave power theorem. The convex proof uses the corresponding convex integrand. The required integral representations are present.

This verdict concerns only these special functions. It does not resolve the separate scope question for arbitrary operator-convex and operator-monotone functions.

3 The Lieb concavity theorem

3.1 The assertion

Let $s \in [0,1]$, let K be any complex matrix, and let $A, B > 0$. Define

$$F_s(A,B) = \operatorname{tr}\bigl(K^\dagger A^s K B^{1-s}\bigr)\,. \tag{12}$$

The formal theorem `lieb_concavity_posDef`, whose input hypotheses use `Matrix.PosDef`, states that F_s is jointly concave on positive-definite pairs. This is the boundary-exponent case of the Ando–Lieb functional in Wolf's Theorem 5.15 [Wol12, Theorem 5.15].

The declaration `lieb_concavity_psd`, whose input hypotheses use `Matrix.PosSemidef`, extends this theorem to positive-semidefinite inputs. The declaration `lieb_concavity_subboundary` then proves the full exponent range

$$x \geq 0, \qquad y \geq 0, \qquad x + y \leq 1 \tag{13}$$

stated in Wolf's Theorem 5.15 [Wol12, Theorem 5.15].

For $r = x + y > 0$, set

$$r = x + y, \qquad s = \frac{x}{r}. \tag{14}$$

Then

$$A^x = (A^r)^s, \tag{15}$$

$$B^y = (B^r)^{1-s}. \tag{16}$$

The proof combines operator concavity of $C \mapsto C^r$, monotonicity of the boundary trace functional, and `lieb_concavity_psd`. When $r = 0$, the functional is constant.

For positive-semidefinite inputs, the proof replaces each matrix by

$$A_{i,\varepsilon} = A_i + \varepsilon \mathbb{1}, \qquad B_{i,\varepsilon} = B_i + \varepsilon \mathbb{1}, \qquad \varepsilon > 0. \tag{17}$$

These matrices are positive definite. Applying the positive-definite theorem and taking $\varepsilon \to 0^+$ proves `lieb_concavity_psd`. The powers converge because $t \mapsto t^s$ is continuous on the nonnegative real axis for $0 < s < 1$, including at zero, and the trace functional is continuous. No logarithmic singularity occurs in this regularization.

3.2 The integral representation

A direct matrix resolvent formula for $A^s B^{1-s}$ is valid only when A and B commute. For noncommuting matrices, $A^s B^{1-s}$ need not be self-adjoint, so that formula cannot prove the general Lieb theorem.

The correct representation uses left and right multiplication operators on $M_D(\mathbb{C})$. Define

$$L_A(X) = AX, \qquad R_B(X) = XB. \tag{18}$$

These operators commute:

$$\begin{aligned} L_A R_B(X) &= AXB \qquad (19)\\ &= R_B L_A(X). \qquad (20) \end{aligned}$$

For $A, B \geq 0$, they are positive operators on the Hilbert–Schmidt space with inner product

$$\langle X, Y \rangle_{\mathrm{HS}} = \mathrm{tr}(X^\dagger Y). \tag{21}$$

The Lieb functional becomes

$$\mathrm{tr}\big(K^\dagger A^s K B^{1-s}\big) = \big\langle K, L_A^s R_B^{1-s} K \big\rangle_{\mathrm{HS}}. \tag{22}$$

For $0 < s < 1$, the commuting-superoperator integral representation is

$$L_A^s R_B^{1-s} = \frac{\sin(\pi s)}{\pi} \int_0^\infty t^{s-1} L_A (L_A + t R_B)^{-1} R_B \, dt. \tag{23}$$

Joint concavity of the integrand in (A, B) for each fixed t follows through the corresponding resolvent argument. Pairing with K in the Hilbert–Schmidt inner product, integrating, and using linearity then proves joint concavity.

Mathlib 4.31 contains neither this superoperator integral representation nor the required matrix-mean and resolvent-monotonicity interface for the commuting pair (L_A, R_B). These ingredients were developed within the formalization. The proof proceeds through the scalar boundary integral, the operator integral representation on the Kronecker model, concavity of the resolvent integrand, and the resulting operator concavity.

3.3 Lieb verdict

The Lieb concavity assertion in Wolf's Theorem 5.15 is correct [Wol12, Theorem 5.15]. The formal proof uses the commuting superoperators L_A and R_B, not a matrix resolvent identity that would require A and B to commute.

The declaration `lieb_concavity_posDef` proves the positive-definite boundary-exponent theorem. The declaration `lieb_concavity_psd` removes the positive-definiteness restriction by regularization, and `lieb_concavity_subboundary` proves the full region in Eq. 13. Three downstream corollaries, including the $K = \mathbb{1}$ case and separate concavity in each argument, build on the positive-definite theorem and are now axiom-free. The former positive-semidefinite and sub-boundary restrictions have both been eliminated.

4 Conclusion

The special-function Jensen theorems and the finite-dimensional Ando–Lieb theorem are proved. The real-power Jensen inequalities retain Wolf's positive subunital hypotheses. The logarithmic clause in Corollary 5.2 of Ref. [Wol12] is false under subunitality and is corrected by requiring unitality. The Lieb theorem now covers positive-semidefinite inputs and all exponents $x, y \geq 0$ with $x + y \leq 1$.

The relevant functional-calculus inputs are `CFC.exists_measure_nnrpow_eq_integral_cfc_rpowIntegrand`, `CFC.concaveOn_rpow`, `CFC.convexOn_rpow`, and `CFC.concaveOn_log`. The three positive-map Jensen inequalities and the Lieb theorem treated in this note are proved.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.