Scope of Wolf Chapter 5 Operator Convexity and Monotonicity Results

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Verdict: scope-restriction (open).

1 The source assertions

Wolf's Section 5.4 develops general results about operator-convex and operator-monotone functions [Wol12, Section 5.4]. All operators below act on finite-dimensional Hilbert spaces. The relation $A \leq B$ denotes the Loewner order. A linear map T is positive if it preserves positive semidefiniteness, subunital if $T(\mathbb{1}) \leq \mathbb{1}$, and unital if $T(\mathbb{1}) = \mathbb{1}$.

Wolf's Theorem 5.8 states that a function $f \colon [0, \infty) \to \mathbb{R}$ is operator convex if and only if it has the representation

$$f(x) = a + bx + cx^2 + \int_0^\infty \frac{yx^2}{y+x}\, d\mu(y), \tag{1}$$

where $a, b \in \mathbb{R}$, $c \geq 0$, and μ is a finite positive measure [Wol12, Theorem 5.8, Equation (5.18)].

Löwner's theorem, stated as Wolf's Theorem 5.9, characterizes continuous operator-monotone functions on $(0, \infty)$ by both their Pick-function continuation and the representation

$$f(x) = a + bx + \int_{-\infty}^0 \frac{1+xy}{y-x}\, d\mu(y), \tag{2}$$

where $a \in \mathbb{R}$, $b \geq 0$, and μ is a finite positive measure [Wol12, Theorem 5.9, Equation (5.19)].

Wolf's Theorem 5.10 assumes, in every dimension, that a Hermitian projection P and a Hermitian operator A satisfy the compression inequality

$$f(PAP) \leq Pf(A)P, \tag{3}$$

whenever the spectrum of A lies in an interval containing zero [Wol12, Theorem 5.10, Equation (5.22)]. It deduces $f(0) \leq 0$, operator convexity of f, and the positive-subunital Jensen inequality

$$f(T(A)) \leq T(f(A)). \tag{4}$$

The last formula is Equation (5.23) of Ref. [Wol12].

Conversely, Wolf's Theorem 5.11 proves for an operator-convex function that

$$f(PAP) \leq Pf(A)P + P^{\perp} f(0) P^{\perp}. \tag{5}$$

This is Equation (5.25) of Ref. [Wol12]. When $f(0) \leq 0$, the same theorem obtains

$$f(T(A)) \leq T(f(A)) \tag{6}$$

for positive subunital T; this is Equation (5.26) of Ref. [Wol12]. Wolf's Theorem 5.12 removes the condition $f(0) \leq 0$ when T is unital [Wol12, Theorem 5.12]. Its entropic corollary is

$$T(\rho) \log T(\rho) \leq T(\rho \log \rho). \tag{7}$$

This is Equation (5.33) of Ref. [Wol12].

For an operator-monotone function on $[0, a]$ satisfying $f(0) \geq 0$, Wolf's Theorem 5.13 gives the reverse Jensen inequality

$$T(f(A)) \leq f(T(A)). \tag{8}$$

This is Equation (5.34) of Ref. [Wol12].

The corollary following Corollary 5.2 extends both Jensen inequalities to a finite family of positive maps $T_1, \ldots, T_n$ satisfying

$$\sum_{i=1}^{n} T_i(\mathbb{1}) \leq \mathbb{1}. \tag{9}$$

For operator-monotone f, it gives

$$\sum_{i=1}^{n} T_i(f(A_i)) \leq f\left(\sum_{i=1}^{n} T_i(A_i)\right). \tag{10}$$

This is Equation (5.37) of Ref. [Wol12]; the operator-convex case gives the reverse inequality.

2 The special-function restriction

The formal development proves Jensen inequalities only for the functions $x \mapsto x^p$ and $x \mapsto \log x$. For a positive subunital map T and a positive semidefinite operator A, it proves

$$T(A^p) \leq (T(A))^p, \qquad 0 \leq p \leq 1, \tag{11}$$

$$(T(A))^p \leq T(A^p), \qquad 1 \leq p \leq 2. \tag{12}$$

For a positive unital map T and a positive-definite operator A, it also proves

$$T(\log A) \leq \log(T(A)). \tag{13}$$

The unital hypothesis in Eq. 13 corrects the false common subunital hypothesis in Wolf's Corollary 5.2 [Wol12, Corollary 5.2].[1]

The formal proofs use integral representations of the specified power functions and derive the logarithmic case by a limit as $p \to 0^+$.[2]

Equations 11, 12, and 13 are instances of Wolf's general Jensen inequalities. They do not imply the integral characterization of every operator-convex function, Löwner's theorem, the projection equivalences, the arbitrary-function Jensen inequalities, or the finite-family corollary.

3 Verdict

The proved power and logarithm inequalities form a proper special-function restriction of the results in Section 5.4 of Ref. [Wol12]. The general operator-convexity and operator-monotonicity assertions remain an open scope gap. No theorem for an arbitrary function f follows from the three special cases alone. The gap is resolved only by proving the general source assertions themselves or source-faithful theorems that imply them.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.

[1] The counterexample and corrected statement are recorded in `docs/paper-gaps/wolf_ch5_operator_jensen_lieb.tex`.

[2] Formal declarations: `posMap_rpow_concave_jensen`, `posMap_rpow_convex_jensen`, and `posMap_log_concave_jensen` in `QICLean/Analysis/LiebConcavity`.