Wolf Chapter 5: Abstract Multiplicative-Domain Hypotheses

2026-07-18

Verdict: clarification (resolved).

1 Source statement

Let $E\colon M_D(\mathbb{C}) \to M_D(\mathbb{C})$ be a complex-linear map. Wolf assumes the one-variable Schwarz inequality

$$E(A^\dagger A) - E(A^\dagger)E(A) \succeq 0 \qquad \text{for every } A \in M_D(\mathbb{C}). \tag{1}$$

This is Equation (5.2) of Ref. [Wol12].[1]

Wolf defines the right and left multiplicative domains by

$$\mathcal{A}_R = \{A \in M_D(\mathbb{C}) : E(XA) = E(X)E(A) \text{ for every } X \in M_D(\mathbb{C})\}\,, \tag{2}$$
$$\mathcal{A}_L = \{A \in M_D(\mathbb{C}) : E(AX) = E(A)E(X) \text{ for every } X \in M_D(\mathbb{C})\}\,. \tag{3}$$

These are Equations (5.11)–(5.12) of Ref. [Wol12].[2]

The corresponding characterizations are

$$A \in \mathcal{A}_R \quad \iff \quad E(A^\dagger A) = E(A^\dagger)E(A), \tag{4}$$
$$A \in \mathcal{A}_L \quad \iff \quad E(AA^\dagger) = E(A)E(A^\dagger). \tag{5}$$

These are Equations (5.13)–(5.14) of Ref. [Wol12]. The theorem assumes complex linearity and Eq. 1; it does not assume a Kraus representation, complete positivity, or 2-positivity.

2 Earlier specialization

The earlier formal characterizations assume a unital Kraus family.[3] These declarations are mathematically correct specializations, but their Kraus and uni-

[1] Local source: `Notes/WolfNoteTexSource/ch05_schwarz_inequalities.tex`, lines 156–164.

[2] Local source: `Notes/WolfNoteTexSource/ch05_schwarz_inequalities.tex`, lines 383–410.

[3] Formal declarations: `KadisonSchwarz.mem_rightMultiplicativeDomain_iff` and `KadisonSchwarz.mem_leftMultiplicativeDomain_iff` in `QICLean/Channel/Schwarz/MultiplicativeDomainFull`.

tality hypotheses do not occur in Wolf's abstract theorem [Wol12, Chapter 5, Equations (5.11)–(5.14)].

The earlier declarations also use a different naming convention. Their subscript names the side on which the varying factor is appended. Wolf's subscript names the side occupied by the fixed element. Consequently, the two conventions interchange the labels "left" and "right", although they describe the same two sets. Their intersections therefore agree:

$$\mathcal{A}_L \cap \mathcal{A}_R = \mathcal{A}_R \cap \mathcal{A}_L. \tag{6}$$

The source-faithful formalization now assumes only complex linearity and the Schwarz inequality in Eq. 1.[4] The Kraus declarations remain available as specialized results. They are no longer used as the blueprint witness for the unrestricted source theorem.

3 Remaining Chapter 5 boundary

Wolf's preceding two-variable operator Schwarz inequality and its equality criterion involve the inverse of $E(B^\dagger B)$ on its range [Wol12, Chapter 5, Theorem 5.3]. They are logically separate from the one-variable result. In particular, the multiplicative-domain theorem does not assume the two-variable theorem. Its polarization argument applies Eq. 1 directly to

$$A + tY, \qquad A, Y \in M_D(\mathbb{C}), \quad t \in \mathbb{C}. \tag{7}$$

The two-variable inverse-on-range inequality is treated separately in `docs/paper-gaps/wolf_ch5_two_variable_unconditional.tex`. Its equality criterion remains separate work and is not needed for the one-variable multiplicative-domain theorem.

4 Verdict

The interchange of the one-sided domain names is a notational clarification. The two conventions assign opposite labels to the same pair of sets, and their full multiplicative domains coincide. The source-faithful one-variable theorem is complete under exactly the abstract hypotheses used by Wolf [Wol12, Chapter 5, Equations (5.2) and (5.11)–(5.14)]. The two-variable inequality and its equality criterion are separate from this resolved clarification.

[4] Formal declarations: `IsSchwarzMap`, `SchwarzMap.rightMultiplicativeDomain`, `SchwarzMap.leftMultiplicativeDomain`, `SchwarzMap.mem_rightMultiplicativeDomain_iff`, and `SchwarzMap.mem_leftMultiplicativeDomain_iff` in `QICLean/Channel/Schwarz/AbstractMultiplicativeDomain`.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.