The Trace Sign in Wolf's Lorentz-Cone Converse

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Verdict: local-correction (resolved).

1 The assertion

Let $A \in M_d(\mathbb{C})$ be Hermitian. Wolf's Proposition 3.7 first states that positive semidefiniteness implies

$$A \geq 0 \quad \Longrightarrow \quad \operatorname{tr}(A^2) \leq \operatorname{tr}(A)^2. \tag{1}$$

The converse printed in the same proposition is

$$\operatorname{tr}(A)^2 \geq (d-1)\operatorname{tr}(A^2) \quad \Longrightarrow \quad A \geq 0. \tag{2}$$

Both assertions appear in Proposition 3.7 of Ref. [Wol12].[1]

2 The point at issue

The squared trace in Eq. 2 does not determine the sign of $\operatorname{tr}(A)$. For $A = -\mathbb{1}_d$ with $d \geq 1$,

$$A = -\mathbb{1}_d \not\geq 0, \tag{3}$$

$$\operatorname{tr}(A)^2 = d^2, \tag{4}$$

$$(d-1)\operatorname{tr}(A^2) = d(d-1), \tag{5}$$

$$\operatorname{tr}(A)^2 \geq (d-1)\operatorname{tr}(A^2). \tag{6}$$

Thus $A = -\mathbb{1}_d$ satisfies the hypothesis of Eq. 2 but contradicts its conclusion. The inequality determines a Lorentz cone without selecting its time orientation; it includes both the positive- and negative-trace sheets.

The issue also appears in Wolf's proof [Wol12, Proposition 3.7]. Write the positive and negative parts of A as A_+ and A_-, so that

$$A = A_+ - A_-, \tag{7}$$

$$\operatorname{tr}(A) = \operatorname{tr}(A_+) - \operatorname{tr}(A_-) \tag{8}$$

$$\leq \operatorname{tr}(A_+). \tag{9}$$

[1] Local source: `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 693–722.

To use the last inequality after squaring, $\mathrm{tr}(A)$ must have the same orientation as the nonnegative quantity $\mathrm{tr}(A_+)$. The condition $\mathrm{tr}(A) \geq 0$ supplies that orientation.

3 Corrected statement

The valid converse adds a nonnegative-trace hypothesis:

$$A = A^\dagger, \qquad \mathrm{tr}(A) \geq 0, \qquad (d-1)\,\mathrm{tr}(A^2) \leq \mathrm{tr}(A)^2 \quad \Longrightarrow \quad A \geq 0. \tag{10}$$

The complex matrix formalization writes the trace conditions using real parts. For Hermitian A, the relevant traces are real, so the formal statement agrees with Eq. 10.[2]

The proof reduces the claim to the eigenvalues of A. Let $\lambda_0, \lambda_1, \ldots, \lambda_{d-1}$ be the eigenvalues, suppose $\lambda_0 < 0$, and define

$$T = \sum_{i=0}^{d-1} \lambda_i, \qquad\qquad R = \sum_{i\neq 0} \lambda_i. \tag{11}$$

The trace assumption gives $T \geq 0$, while $T = \lambda_0 + R$ and $\lambda_0 < 0$ imply

$$0 \leq T < R, \tag{12}$$

$$T^2 < R^2. \tag{13}$$

Cauchy's inequality for the remaining $d-1$ eigenvalues yields

$$R^2 \leq (d-1) \sum_{i\neq 0} \lambda_i^2 \tag{14}$$

$$\leq (d-1) \sum_{i=0}^{d-1} \lambda_i^2. \tag{15}$$

Because $T = \mathrm{tr}(A)$ and $\sum_i \lambda_i^2 = \mathrm{tr}(A^2)$, combining Eqs. 13 and 15 gives

$$\mathrm{tr}(A)^2 < (d-1)\,\mathrm{tr}(A^2), \tag{16}$$

contrary to the hypothesis in Eq. 10. Hence no eigenvalue is negative, and $A \geq 0$.

4 Conclusion

The converse in Proposition 3.7 of Ref. [Wol12] is false without a trace-sign condition. The local correction is Eq. 10. The formal development records this trace-nonnegative, future-cone statement for use in the positive-map discussion and does not mark the unrestricted printed converse as formalized.

[2] Formal declaration: `Matrix.IsHermitian.posSemidef_of_trace_re_nonneg_of_card_sub_one_mul_trace_sq_re_le` in `QICLean/Analysis/MatrixTraceInequalities`. Tracked by TNLean issue #3402.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.