# The Nonvanishing Partner in Wolf's Kramers Theorem II

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**Verdict:** false-source (resolved).

## 1 The assertion

Wolf's Kramers theorem II concerns a Hermitian matrix $H$ and a nonzero anti-symmetric matrix $A$ satisfying an intertwining relation. It asserts that

$$H = H^\dagger, \quad A^T = -A, \quad A \neq 0, \quad HA = AH^T \quad \Longrightarrow \quad \dim\ker(H - \lambda\mathbb{1}) \geq 2 \text{ for every eigenvalue } \lambda \text{ of } H. \tag{1}$$

This is the statement printed as Chapter 3, Theorem 3.2, "Kramer's theorem II", in Ref. [Wol12].[1]

The theorem is presented as a relaxation of Wolf's first Kramers theorem [Wol12, Chapter 3, Theorem 3.1]. That theorem assumes that a Hermitian matrix $H$ commutes with an anti-unitary operator $T$ satisfying $T^2 = -\mathbb{1}$. Writing $T = \Gamma V$, where $\Gamma$ is complex conjugation and $V$ is unitary, gives

$$HV^\dagger = V^\dagger H^T, \qquad V^T = -V. \tag{2}$$

Wolf then states that "the restriction on the required symmetry in the theorem can be relaxed as the same proof shows" [Wol12, Chapter 3, Theorems 3.1–3.2].[2]

## 2 The point at issue

Let $|\psi\rangle \neq 0$ satisfy

$$H|\psi\rangle = \lambda|\psi\rangle. \tag{3}$$

[1] Local source: `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 527–530. Tracked in GitHub issue #17.

[2] Local source: lines 498–501 for the matrix reduction and lines 523–525 for the relaxation claim.

Because $H$ is Hermitian, $\lambda$ is real. Complex conjugation of Eq. 3 therefore gives

$$H^T|\overline{\psi}\rangle = \overline{H|\psi\rangle} \tag{4}$$

$$= \lambda|\overline{\psi}\rangle. \tag{5}$$

The intertwining relation then yields

$$HA|\overline{\psi}\rangle = AH^T|\overline{\psi}\rangle \tag{6}$$

$$= \lambda A|\overline{\psi}\rangle. \tag{7}$$

Thus $A|\overline{\psi}\rangle$ is either zero or an eigenvector of $H$ with eigenvalue $\lambda$.

Anti-symmetry also gives

$$\langle\psi|A\overline{\psi}\rangle = \overline{\psi}^T A\overline{\psi} \tag{8}$$

$$= \overline{\psi}^T A^T\overline{\psi} \tag{9}$$

$$= -\overline{\psi}^T A\overline{\psi} \tag{10}$$

$$= 0. \tag{11}$$

The second equality uses the fact that a scalar equals its transpose. Equations 7 and 11 establish the transport and orthogonality steps in the printed proof. They do not establish that the partner is nonzero. In the first Kramers theorem, the partner is $V^\dagger|\overline{\psi}\rangle$, which cannot vanish because $V$ is unitary. A general nonzero anti-symmetric matrix $A$ may have a nontrivial kernel.

A counterexample occurs on $\mathbb{C}^3$. Define

$$H = \mathrm{diag}(0,1,1), \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{12}$$

These matrices satisfy

$$H = H^\dagger, \qquad A^T = -A, \qquad A \neq 0, \tag{13}$$

$$HA = AH^T = A. \tag{14}$$

The nonzero rows and columns of $A$ lie in the 1-eigenspace of $H$, which explains Eq. 14. The eigenvalue 0 is simple, with eigenvector $e_1$, and its proposed partner vanishes:

$$He_1 = 0, \qquad \dim\ker H = 1, \qquad A\overline{e_1} = Ae_1 = 0. \tag{15}$$

Thus the hypotheses of Eq. 1 hold while its conclusion fails.

# 3 The corrected statement

For an eigenvector $|\psi\rangle$ with eigenvalue $\lambda$, the calculation proves the exact conditional statement

$$A|\overline{\psi}\rangle \neq 0 \quad \Longrightarrow \quad \dim\ker(H - \lambda\mathbb{1}) \geq 2. \tag{16}$$

Under the hypothesis in Eq. 16, Eqs. 7 and 11 provide two nonzero orthogonal eigenvectors with eigenvalue $\lambda$.

A global sufficient condition is injectivity of $A$. More precisely,

$$H = H^\dagger, \qquad A^T = -A, \quad HA = AH^T, \quad \ker A = \{0\}, \tag{17}$$

$$\dim\ker(H - \lambda\mathbb{1}) \geq 2 \qquad \text{for every eigenvalue } \lambda \text{ of } H. \tag{18}$$

Here the first line implies the second. In finite dimensions, $\ker A = \{0\}$ is equivalent to invertibility.

If $A$ is a $d \times d$ anti-symmetric matrix, then

$$\det A = \det A^T \tag{19}$$

$$= \det(-A) \tag{20}$$

$$= (-1)^d \det A. \tag{21}$$

Consequently, an invertible anti-symmetric matrix can exist only in even dimension. This agrees with Wolf's observation that anti-symmetric unitaries exist only in even dimensions [Wol12, Chapter 3, discussion following Theorem 3.1].[3]

The first Kramers theorem is the invertible case of Eqs. 17–18. For $T = \Gamma V$, set $A = V^\dagger$. The relation $V^T = -V$ and unitarity imply $A^T = -A$, while unitarity also makes $A$ invertible. The corrected theorem II therefore specializes to theorem I using Wolf's proof [Wol12, Chapter 3, Theorems 3.1–3.2].

# 4 The formalization

The corrected development records transport, orthogonality, the conditional theorem, the invertible global corollary, and both forms of the first Kramers theorem.[4] The formal statements express degeneracy by the lower bound $2 \leq \dim$ on the eigenspace. For Hermitian matrices, geometric and algebraic multiplicities coincide, so this agrees with the printed phrase "two-fold degenerate". The false statement in Eq. 1 is deliberately not formalized.

# 5 Conclusion

Kramers theorem II is false as printed in Ref. [Wol12]. The condition $A \neq 0$ does not prevent the partner $A|\overline{\psi}\rangle$ from vanishing. The exact local conclusion

[3] Local source, lines 523–524.

[4] Formal development: `QICLean/Algebra/KramersDegeneracy`. Transport: `Matrix.IsHermitian.mulVec_star_intertwiner_of_mul_eq_mul_transpose`; orthogonality: `Matrix.dotProduct_star_mulVec_star_eq_zero_of_transpose_eq_neg`; conditional theorem: `Matrix.IsHermitian.two_le_finrank_eigenspace_of_intertwiner_mulVec_star_ne_zero`; invertible global corollary: `Matrix.IsHermitian.two_le_finrank_eigenspace_of_antisymmetric_isUnit`; theorem I in matrix form, obtained with $A = V^\dagger$: `Matrix.IsHermitian.two_le_finrank_eigenspace_of_antisymmetric_unitary`; theorem I under the printed anti-unitary hypotheses: `Matrix.IsHermitian.two_le_finrank_eigenspace_of_antiunitary`.

is Eq. 16, and the global degeneracy conclusion follows under the invertibility condition in Eq. 17. These corrected statements resolve the source error.

## References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.