

The Rotation Sign in Wolf's Spinor Exponential Formula

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Verdict: false-source (resolved).

1 The printed formula

Let $\sigma_1, \sigma_2, \sigma_3$ be the standard Pauli matrices. Wolf defines the spatial rotation generators by

$$R_i = \sum_{j,k=1}^3 \varepsilon_{ijk} |k\rangle\langle j|. \quad (1)$$

For $\vec{n} \in \mathbb{R}^3$, Equation (2.44) of Ref. [Wol12] prints the correspondence

$$U = \exp\left(-\frac{i}{2} \vec{n} \cdot \vec{\sigma}\right), \quad (2)$$

$$U \mapsto \exp\left(-\vec{n} \cdot \vec{R}\right). \quad (3)$$

The local source is `Notes/WolfNoteTexSource/ch02_representations.tex`, lines 1070–1077.

2 The sign inconsistency

Take

$$\vec{n} = (0, 0, \theta). \quad (4)$$

With the standard matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (5)$$

and the column-vector convention used in the source, conjugation gives

$$e^{-i\theta\sigma_3/2} \sigma_1 e^{i\theta\sigma_3/2} = \cos \theta \sigma_1 + \sin \theta \sigma_2. \quad (6)$$

Thus the Pauli-coordinate vector of σ_1 rotates toward the positive second coordinate.

By (1), the third generator satisfies

$$R_3|1\rangle = |2\rangle. \quad (7)$$

Therefore

$$e^{+\theta R_3}|1\rangle = \cos \theta |1\rangle + \sin \theta |2\rangle, \quad (8)$$

$$e^{-\theta R_3}|1\rangle = \cos \theta |1\rangle - \sin \theta |2\rangle. \quad (9)$$

The conjugation in (6) agrees with (8), not with (9). Hence the two negative signs in the printed correspondence cannot both be correct.

3 Correction and formalization

For the spinor in (2) and the generators in (1), the corrected congruence action is

$$\exp\left(-\frac{i}{2}\vec{n} \cdot \vec{\sigma}\right) \mapsto \exp\left(+\vec{n} \cdot \vec{R}\right). \quad (10)$$

Equivalently, the negative Lorentz exponential printed in Equation (2.44) of Ref. [Wol12] corresponds to

$$U = \exp\left(+\frac{i}{2}\vec{n} \cdot \vec{\sigma}\right). \quad (11)$$

The corrected relation (10) is formalized directly. The formal development also contains the closed spinor exponential, Rodrigues' formula, and the consistent positive-angle diagonal-SU(2) calculation.¹

4 Verdict

False source. Equation (2.44) of Ref. [Wol12] contains a genuine sign error. This is not a convention change introduced by the formalization: the source explicitly fixes the Pauli matrices, the generators in (1), and the column-vector action. The boost part of Equation (2.44) is unaffected.

References

- [Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.

¹Formal declarations: `Wolf.spinorMatrix_rotationExpSL2`, `Wolf.exp_neg_I_smul_pauliVector`, and `Wolf.exp_smul_rotationGenerator` in `QIClean/Channel/LorentzNormalForm/SpinorExponential`; the positive-angle diagonal calculation is `SpinCover.R_su2Diag_eq_rotZ`.