Zero Multiplicities in Wolf's Two-Pure-State Spectrum Shorthand

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Verdict: local-correction (resolved).

1 The source display

Let $P, Q \in M_d(\mathbb{C})$ be Hermitian rank-one projections. In the spectrum-preserver argument following Wigner's theorem, Wolf writes the spectrum of $P+Q$ as

$$1-\sqrt{\operatorname{tr}(PQ)}, \qquad 1+\sqrt{\operatorname{tr}(PQ)}. \tag{1}$$

This shorthand appears in Chapter 1, lines 289–296 of Ref. [Wol12]; the local source is `Notes/WolfNoteTexSource/ch01_deconstructing_quantum.tex`.

2 The suppressed eigenvalues

The range of $P+Q$ lies in the span of the one-dimensional ranges of P and Q. Hence $P+Q$ vanishes on a subspace of dimension at least $d-2$. For $d \geq 2$, its characteristic polynomial, with algebraic multiplicities, is

$$\chi_{P+Q}(X) = X^{d-2}((X-1)^2 - \operatorname{tr}(PQ)). \tag{2}$$

The two values in (1) are the roots of the quadratic factor in (2). The factor X^{d-2} contributes at least $d-2$ zero eigenvalues.

If $P = Q$, then $\operatorname{tr}(PQ) = 1$, so

$$(X-1)^2 - \operatorname{tr}(PQ) = X(X-2). \tag{3}$$

The quadratic factor then contributes one additional zero, and the total zero multiplicity is $d-1$.

In dimension one, every normalized rank-one projection is the identity, so the separate formula is

$$\chi_{P+Q}(X) = X-2. \tag{4}$$

In dimension zero, projective pure states do not exist.

3 Formal correction

The formalization proves the multiplicity-aware identity (2) through a $d \times 2$ and $2 \times d$ factorization. It then recovers $\mathrm{tr}(PQ)$ from equality of two such characteristic polynomials.[1] The low-dimensional cases are stated explicitly in the same file. This correction is tracked by TNLean issue #5683.

4 Classification

Local correction. The issue is a spectrum shorthand, not a failure of Wolf's intended spectrum-preserver argument [Wol12]. The two roots in (1) retain the transition-probability information used in that argument. The formal theorem uses characteristic polynomials because they record every zero eigenvalue and every algebraic multiplicity without ambiguity.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.

[1] Formal declarations: `Projectivization.charpoly_add_pureStateMatrix_of_two_le` and `Projectivization.trace_pureStateMatrix_mul_eq_of_charpoly_add_eq` in `QICLean/Channel/Wigner/TwoPureStateCharpoly`.