Wolf's Operator-System Extension: Matrix-Algebra Realization

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Verdict: scope-restriction (resolved).

1 The source statement

Wolf's proposition "Extending cp maps from operator systems" concerns arbitrary finite-dimensional C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ [Wol12, lines 614–626]. Let $S \subseteq \mathcal{A}$ be an operator system, and let $T : S \to \mathcal{B}$ be completely positive. The proposition asserts that there is a completely positive map $\widetilde{T} : \mathcal{A} \to \mathcal{B}$ satisfying

$$\widetilde{T}(x) = T(x) \qquad (x \in S). \tag{1}$$

The local source is `Notes/WolfNoteTexSource/ch01_deconstructing_quantum.tex`, lines 617–619 for the statement and lines 620–626 for the proof.

Neither $\mathcal{A}$ nor $\mathcal{B}$ is assumed to be a full matrix algebra. By the finite-dimensional structure theorem, a finite-dimensional C^*-algebra is a finite direct sum of full matrix algebras:

$$\mathcal{A} \cong \bigoplus_k M_{d_k}(\mathbb{C}). \tag{2}$$

Thus Wolf's proposition covers all such direct sums, not only the single-summand case $\mathcal{A} = M_m(\mathbb{C})$.

2 The formalized domain

The formalized domain has the full matrix-realized generality stated in the source:

$$\mathcal{A} = \bigoplus_{k<r} M_{d_k}(\mathbb{C}). \tag{3}$$

It is represented literally by the dependent function type $\forall k, M_{d_k}(\mathbb{C})$. The predicates `Matrix.IsOperatorSystemDirectSum` and `Matrix.IsCPOnOpera` torSystemDirectSum define operator systems and complete positivity at every matrix level block by block. When $r = 1$, they agree verbatim with the single-summand predicates Matrix.IsOperatorSystem and Matrix.IsCPOnOperatorSystem.[1]

For a full matrix codomain $M_p(\mathbb{C})$, the extension theorem produces

$$\widetilde{T} : \bigoplus_k M_{d_k}(\mathbb{C}) \longrightarrow M_p(\mathbb{C}), \tag{4}$$

$$\widetilde{T}|_S = T. \tag{5}$$

Complete positivity is expressed by the blockwise predicate Matrix.IsCPMapDirectSum.[2]

The proof transports the single-summand extension theorem along the block-diagonal embedding

$$\bigoplus_k M_{d_k}(\mathbb{C}) \hookrightarrow M_M(\mathbb{C}), \tag{6}$$

$$M = \sum_k d_k. \tag{7}$$

The embedding is Matrix.directSumEmbedFin, constructed from Matrix.directSumDiagonalEmbedding and the canonical reindexing finSigmaFinEquiv. It is the same block-diagonal construction used for the block structure in Wolf's Theorem 6.14 [Wol12]. Complete positivity at every tensor level descends through the embedding because a block-diagonal matrix is positive semidefinite exactly when each diagonal block is positive semidefinite.[3]

3 The formalized codomain

The codomain may be any concrete unital $*$-subalgebra

$$\mathcal{B} \subseteq M_p(\mathbb{C}). \tag{8}$$

Given $T : S \to \mathcal{B}$, the formal theorem produces

$$\widetilde{T} : \bigoplus_k M_{d_k}(\mathbb{C}) \longrightarrow \mathcal{B}, \tag{9}$$

$$\widetilde{T}|_S = T. \tag{10}$$

[1]Direct-sum definitions: QICLean/Channel/OperatorSystemExtensionDirectSum. Single-summand definitions: QICLean/Channel/OperatorSystem.

[2]Formal declaration: Matrix.exists_cp_extension_of_operatorSystem_directSum in QICLean/Channel/OperatorSystemExtensionDirectSum.

[3]Single-summand extension: Matrix.exists_cp_extension_of_operatorSystem in QICLean/Channel/OperatorSystemExtension. Tensor-level descent: Matrix.bipartiteSlice_directSumEmbedFinLevel.

Complete positivity is stated after the canonical inclusion $\mathcal{B} \hookrightarrow M_p(\mathbb{C})$, because the direct-sum complete-positivity predicate uses a full matrix codomain.[4]

The construction supplies the final step left implicit in Wolf's proof [Wol12, lines 620–626]. First, the direct-sum theorem gives an ambient completely positive extension

$$F : \bigoplus_k M_{d_k}(\mathbb{C}) \longrightarrow M_p(\mathbb{C}). \tag{11}$$

Next, a trace-preserving completely positive conditional expectation

$$E : M_p(\mathbb{C}) \longrightarrow M_p(\mathbb{C}) \tag{12}$$

has range in $\mathcal{B}$ and restricts to the identity on $\mathcal{B}$. Therefore $E \circ F$ is completely positive, takes values in $\mathcal{B}$, and agrees with T on S. Restricting its codomain gives (9).[5]

4 Verdict

Closed for matrix-realized finite-dimensional algebras. The domain is a finite direct sum $\bigoplus_{k<r} M_{d_k}(\mathbb{C})$, and the codomain is a concrete unital $*$-subalgebra of a full matrix algebra. This is the representation used in Wolf's displayed proof [Wol12, lines 620–626]. The formal theorem does not quantify over an abstract finite-dimensional C^*-algebra type without a specified matrix representation and must not be cited as such an abstract theorem.

This result closes the matrix-algebra scope gap tracked in TNLean issue #5645. Neither the domain nor the codomain is required to be a single full matrix algebra.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.

[4]Formal declaration: `Matrix.exists_cp_extension_of_operatorSystem_directSum_starSubalgebra` in `QICLean/Channel/OperatorSystemExtensionStarSubalgebra`.

[5]Formal declaration: `StarSubalgebra.exists_krausCPTP_conditionalExpectation`.