The General Compact-Convex Brouwer Fixed-Point Theorem

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Verdict: open-gap (resolved).

1 The assertion

Let $S \subseteq \mathbb{R}^n$ be nonempty, compact, and convex, and let $T : S \to S$ be continuous. Wolf's Theorem 6.10 states the general compact-convex Brouwer fixed-point theorem:

$$\exists\, x \in S \quad \text{such that} \quad T(x) = x. \tag{1}$$

See Ref. [Wol12, Theorem 6.10].[1]

2 Resolution

The vendored Gametheory library, LionSR/Brouwer, proves the Brouwer fixed-point theorem for the standard simplex Δ^n.[2] QICLean extends that theorem to products of simplices, closed cubes, compact retracts of finite-dimensional real normed spaces, and arbitrary nonempty compact convex subsets.[3]

The last two declarations prove, respectively, a coordinate-free theorem and Wolf's printed theorem for arbitrary nonempty compact convex $S \subseteq \mathbb{R}^n$. The proof uses metric projection. For each ambient point, compactness supplies a minimizer of the squared distance over S. Convexity gives the first-order variational inequality, from which the chosen projection is 1-Lipschitz. The projection restricts to the identity on S, so it is the continuous retraction required by the compact-retract fixed-point theorem.[4]

[1] Local source: `Notes/WolfNoteTexSource/ch06\_spectral\_properties.tex`, lines 1143–1151.

[2] Formal declaration: `Brouwer`.

[3] Formal declarations: `exists_fixedPoint_closedCube` in `QICLean/Topology/BrouwerProduct`, `fixedPoint_of_compact_retract` in `QICLean/Topology/CompactRetractFixedPoint`, and `fixedPoint_of_compact_convex` and `brouwer_fixedPoint_compactConvex` in `QICLean/Topology/CompactConvexFixedPoint`.

[4] Reusable formal declarations: `CompactConvex.metricProjection`, `CompactConvex.nearest_inner_nonpos`, and `CompactConvex.metricProjection_lipschitzWith`.

Wolf gives only the theorem statement at the cited location [Wol12, Theorem 6.10]. The metric-projection argument is the formal proof and is not attributed to the lecture notes.

3 Impact

The former gap in the general statement did not block the formalization of Wolf's Theorem 6.11 on stationary states, Proposition 6.8 on positive fixed points, or Corollary 6.5 on spanning by stationary density matrices [Wol12]. Those results require only the density-matrix form of Brouwer, which was already proved. The channel results continue to use that specialization; the general compact-convex theorem is now available independently.

4 Verdict

Resolved. Wolf's general compact-convex statement (1) is formalized as `brouwer_fixedPoint_compactConvex`. The coordinate-free theorem and the reusable nonexpansive metric projection are formalized with it.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.