Hypothesis Boundaries for the Indecomposability of the Breuer–Hall Map (Wolf Example 3.1)

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Verdict: local-correction (resolved).

1 Source statement

Let $U \in M_D(\mathbb{C})$ be antisymmetric, so $U^\intercal = -U$, and suppose that it is a contraction:

$$U^\dagger U \leq \mathbb{1}. \tag{1}$$

Wolf's Chapter 3, Example 3.1 defines the Breuer–Hall map by

$$T_{\mathrm{BH}}(X) = \mathrm{tr}(X)\mathbb{1} - X - UX^\intercal U^\dagger. \tag{2}$$

The source identifies this formula as Equation (3.19) [Wol12, Chapter 3, Example 3.1].[1]

Wolf states that antisymmetric unitaries exist when d is even and satisfy "$U^\dagger = U^{-1} = -\overline{U}$". The source then states, without an explicit dimension hypothesis, that T_{BH} is not decomposable when U is an antisymmetric unitary [Wol12, Example 3.1]. Its positivity and indecomposability assertions therefore use different hypotheses: positivity allows the contraction condition (1), whereas indecomposability requires

$$U^\dagger U = \mathbb{1}. \tag{3}$$

The formalization preserves this distinction. It proves positivity for antisymmetric contractions and indecomposability for antisymmetric unitaries when $d > 2$. Indecomposability does not extend to all antisymmetric contractions.

2 The dimension-two boundary

Every antisymmetric 2×2 matrix is a scalar multiple of

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

[1] Local source: `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 344–355. Formalization issue: GitHub issue #16.

Thus every antisymmetric unitary in dimension two has the form $e^{i\theta}J$, and the phase does not affect the twisted-transpose term. For

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5}$$

direct multiplication gives

$$JX^{\mathsf{T}}J^{\dagger} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{6}$$

$$JX^{\mathsf{T}}J^{\dagger} = \operatorname{tr}(X)\mathbb{1} - X. \tag{7}$$

Substitution into (2) yields

$$T_{\mathrm{BH}}(X) = 0 \tag{8}$$

for every $X \in M_2(\mathbb{C})$. The zero map is completely positive and therefore decomposable. Hence $d > 2$ is necessary.

An antisymmetric unitary can exist only in even dimension. Indeed, $U^{\mathsf{T}} = -U$ implies

$$\det U = \det(U^{\mathsf{T}}), \tag{9}$$

$$\det U = \det(-U), \tag{10}$$

$$\det U = (-1)^d \det U. \tag{11}$$

Since unitarity gives $\det U \neq 0$, the dimension is even. The corrected unitary range is therefore $d \geq 4$.

3 Why contractivity does not imply indecomposability

The zero matrix is an antisymmetric contraction in every dimension:

$$0^{\mathsf{T}} = -0, \tag{12}$$

$$0^{\dagger}0 \leq \mathbb{1}. \tag{13}$$

At $U = 0$, the Breuer–Hall map becomes the first reduction map:

$$T_{\mathrm{BH},0}(X) = \operatorname{tr}(X)\mathbb{1} - X, \tag{14}$$

$$T_{\mathrm{BH},0}(X) = T_1(X). \tag{15}$$

Let θ denote transposition and F the swap operator. With the normalization used in the formal development, the Choi matrix of $T_1 \circ \theta$ is

$$\tau(T_1 \circ \theta) = d^{-1}(\mathbb{1} - F), \tag{16}$$

$$\tau(T_1 \circ \theta) \geq 0. \tag{17}$$

The second line holds because $\mathbb{1} - F$ is twice the orthogonal projector onto the antisymmetric subspace. The Choi criterion shows that $T_1 \circ \theta$ is completely positive. Therefore T_1 is completely copositive and decomposable.

Consequently, even when $d > 2$, the Breuer–Hall map need not be indecomposable over all antisymmetric contractions. This counterexample does not classify the nonunitary contractions for which the map is indecomposable, and the formalization makes no such claim.[2]

4 The corrected statement

If $d > 2$ and U is an antisymmetric unitary on $\mathbb{C}^d$, then T_{BH} is an indecomposable positive map.[3]

The source gives no proof. The formalized argument uses PPT detection. Let F be the swap operator on $\mathbb{C}^d \otimes \mathbb{C}^d$, and let $|\Psi\rangle$ be the U-twisted maximally entangled vector with coordinates $\Psi(i,k) = U(i,k)$. Define the unnormalized projector and witness state by

$$P_0 = |\Psi\rangle\langle\Psi|, \tag{18}$$

$$\rho = (\mathbb{1} + F) + P_0. \tag{19}$$

Both terms in (19) are positive semidefinite. Let P_1 denote the untwisted maximally entangled projector. Antisymmetry and unitarity give

$$(\mathbb{1} + F)^{T_1} = \mathbb{1} + P_1, \tag{20}$$

$$P_0^{T_1} = (U^\dagger \otimes \mathbb{1})(\mathbb{1} + F)(U \otimes \mathbb{1}) - \mathbb{1}. \tag{21}$$

Therefore

$$\rho^{T_1} = P_1 + (U^\dagger \otimes \mathbb{1})(\mathbb{1} + F)(U \otimes \mathbb{1}), \tag{22}$$

$$\rho^{T_1} \geq 0. \tag{23}$$

Thus ρ is a PPT state.

Expanding the tensor indices and using $U^\dagger U = \mathbb{1}$ gives, for every bipartite matrix σ,

$$\mathrm{tr}[(T_{\mathrm{BH}} \otimes \mathrm{id})(\sigma) P_0] = \mathrm{tr}(\sigma) - \mathrm{tr}(\sigma P_0) - \mathrm{tr}(F\sigma). \tag{24}$$

[2] Formal declarations: `Matrix.breuerHallMap_zero` in `QICLean/Channel/BreuerHallMap`; `ChoiJamiolkowski.choiMatrix_reductionMap_one_comp_transpose`, `Matrix.reductionMap_one_isCompletelyCopositiveMap`, and `Matrix.reductionMap_one_isDecomposablePositiveMap` in `QICLean/Channel/PositiveExamples`; and `Matrix.breuerHallMap_contraction_not_universally_indecomposable` in `QICLean/Channel/BreuerHallIndecomposable`.

[3] Formal declarations: `Matrix.breuerHallMap_isIndecomposablePositiveMap` and `Matrix.breuerHallMap_not_isDecomposablePositiveMap` in `QICLean/Channel/BreuerHallIndecomposable`; positivity is `Matrix.breuerHallMap_isPositiveMap` in `QICLean/Channel/BreuerHallMap`.

For $\sigma = \rho$, antisymmetry gives $F|\Psi\rangle = -|\Psi\rangle$, and the required traces are

$$\operatorname{tr}(\rho) = d^2 + 2d, \tag{25}$$
$$\operatorname{tr}(\rho P_0) = d^2, \tag{26}$$
$$\operatorname{tr}(F\rho) = d^2. \tag{27}$$

Substitution into (24) yields

$$\operatorname{tr}[(T_{\mathrm{BH}} \otimes \mathrm{id})(\rho)P_0] = d(2-d), \tag{28}$$
$$\operatorname{tr}[(T_{\mathrm{BH}} \otimes \mathrm{id})(\rho)P_0] < 0 \qquad (d > 2). \tag{29}$$

Hence $(T_{\mathrm{BH}} \otimes \mathrm{id})(\rho)$ is not positive semidefinite. Decomposable maps preserve positivity on PPT states.[4] Therefore T_{BH} is not decomposable.

5 Conclusion

Wolf's positivity statement applies to antisymmetric contractions, while the indecomposability statement applies only to antisymmetric unitaries [Wol12, Example 3.1]. In the unitary case, the dimension correction is necessary: the formal theorem assumes $d > 2$, equivalently even $d \geq 4$ when an antisymmetric unitary exists, and the map vanishes at $d = 2$. The unitary hypothesis is also a genuine boundary for the stated conclusion, because the antisymmetric contraction $U = 0$ gives the completely copositive, decomposable reduction map T_1. The formalization records the corrected unitary theorem and this counterexample without asserting a classification for general contractions.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.

[4] Formal declarations: `IsDecomposablePositiveMap.tensorMapId_posSemidef_of_isPPT` and `not_isDecomposablePositiveMap_of_isPPT_not_tensorMapId_posSemidef` in `QICLean/Channel/DecomposablePPT`.