

The Structure Theorem for Finite-Dimensional C^* -Algebras: Matrix-Realized Single-Block Scope

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Verdict: scope-restriction (resolved).

1 The source statement

Schumacher and Werner state the structure theorem for finite-dimensional C^* -algebras as their Proposition “Csform” [SW04, lines 2082–2098]: every finite-dimensional C^* -algebra $\mathcal{A}$ is determined up to isomorphism by a finite sequence $n_1 \geq n_2 \geq \dots \geq 1$ with

$$\mathcal{A} \cong \bigoplus_{\mu} M_{n_{\mu}}(\mathbb{C}). \quad (1)$$

The proof sketch given there cuts $\mathcal{A}$ by the minimal projections z_{μ} of its centre,

$$\mathcal{A} = \bigoplus_{\mu} z_{\mu} \mathcal{A}, \quad (2)$$

and asserts that each summand $z_{\mu} \mathcal{A}$, whose centre consists of the multiples of its unit, is isomorphic to a full matrix algebra $M_{n_{\mu}}(\mathbb{C})$.

Gross, Nesme, Vogts and Werner use exactly this building-block assertion, not (1) itself: a support algebra $\mathcal{R}_x$ with trivial centre “must be isomorphic to $M_{r(x)}$ ” [GNVW12, lines 1278–1282].

2 The point at issue

Two features of (1) are outside the formalized statement.

First, $\mathcal{A}$ is an *abstract* finite-dimensional C^* -algebra. The formalized statement quantifies over a $*$ -subalgebra

$$S \subseteq M_D(\mathbb{C}), \quad (3)$$

that is, over an algebra already presented with a faithful finite-dimensional $*$ -representation. Every finite-dimensional C^* -algebra admits such a presentation,

but the passage from the abstract algebra to (3) is the Gelfand–Naimark theorem, which is not formalized here.

Second, the formalized statement is the single-block case. The centre of S is assumed to be the scalar image,

$$Z(S) = \mathbb{C} \cdot 1, \quad (4)$$

which is the hypothesis satisfied by one summand $z_\mu \mathcal{A}$ of (2). Under (4) the conclusion is a single full matrix algebra,

$$S \cong M_r(\mathbb{C}), \quad r \geq 1, \quad (5)$$

as a $*$ -isomorphism.¹ The decomposition (2) into minimal central projections, and with it the passage from (5) to (1), is not formalized.

3 The formalized route

The formal proof does not follow the minimal-projection sketch. It uses the Wedderburn–Artin decomposition of S as an abstract semisimple ring over the algebraically closed field $\mathbb{C}$,

$$S \cong \prod_{k=1}^m M_{d_k}(\mathbb{C}), \quad d_k \geq 1, \quad (6)$$

which is available because a $*$ -subalgebra of $M_D(\mathbb{C})$ has vanishing Jacobson radical.² Under (4) the coordinate idempotent of the k -th factor of (6) is central, hence a scalar multiple of the unit; a scalar that is the unit in one coordinate and zero in another forces $m \leq 1$, and $m = 0$ is excluded because S contains the unit of $M_D(\mathbb{C})$. So $m = 1$ and S is a simple ring, and the Wedderburn theorem for simple algebras over $\mathbb{C}$ gives an algebra isomorphism

$$\varphi : S \longrightarrow M_r(\mathbb{C}). \quad (7)$$

The isomorphism (7) is supplied by ring theory alone and carries no compatibility with the adjoint. The formal development therefore adds the step that the source leaves inside its phrase “are then seen to be isomorphic”: transporting the adjoint of S through φ gives an algebra automorphism of $M_r(\mathbb{C})$, which by Skolem–Noether is conjugation by an invertible H , so that

$$\varphi(y^\dagger) = H^{-1} \varphi(y)^\dagger H, \quad y \in S. \quad (8)$$

Applying (8) twice shows that $H^\dagger$ is a scalar multiple of H . Evaluating the trace of φ^{-1} on the rank-one matrices $Y = vw^\dagger$, for which

$$Y^\dagger H Y = (v^\dagger H v) w w^\dagger, \quad (9)$$

¹Formal declaration: `StarSubalgebra.exists_starAlgEquiv_matrix_of_isCentral` in `QIClean/Algebra/StarSubalgebraFactor`.

²Formal declarations: `StarSubalgebra.isSemisimpleRing` and `StarSubalgebra.exists_algEquiv_pi_matrix` in `QIClean/Algebra/StarSubalgebraSemisimple`.

shows that $v^\dagger H v$ never vanishes for $v \neq 0$ and that, after H is rescaled by one scalar, the Hermitian form has constant sign. The rescaled matrix is positive definite; writing it as $R^\dagger R$ with R invertible, the corrected map

$$y \longmapsto R \varphi(y) R^{-1} \tag{10}$$

preserves adjoints and realizes (5).³

4 Verdict

Closed for the matrix-realized single-block case. The formal statement proves (5) for a $*$ -subalgebra of a nonzero full matrix algebra whose centre is scalar, which is the assertion used in [GNVW12, lines 1278–1282]. It must not be cited as (1): neither the abstract C^* -algebra generality nor the decomposition (2) into minimal central projections is formalized. Removing the restriction requires the minimal central projections of S and a compatible reassembly of the summands; the building block proved here is the step that reassembly would repeat for each μ .⁴

References

- [GNVW12] David Gross, Vincent Nesme, Holger Vogts, and Reinhard F. Werner. Index theory of one dimensional quantum walks and cellular automata. *Communications in Mathematical Physics*, 310(2):419–454, 2012.
- [SW04] Benjamin Schumacher and Reinhard F. Werner. Reversible quantum cellular automata. arXiv:quant-ph/0405174, 2004.

³Formal declaration: `StarSubalgebra.exists_starAlgEquiv_of_algEquiv_matrix`.

⁴Consumer: GitHub issue #463.