The Minkowski Canonical-Pair Dependency in Four Real Dimensions

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Verdict: open-gap (wip).

1 The required assertion

Define the Minkowski metric

$$M = \operatorname{diag}(1, -1, -1, -1). \tag{1}$$

Let $C \in M_4(\mathbb{R})$ be self-adjoint with respect to M, meaning that

$$C^{\mathsf{T}} M = MC. \tag{2}$$

The proof of Theorem 3 of Verstraete–Dehaene–De Moor invokes Gohberg–Lancaster–Rodman, Part I, Chapter 5, Section 5.1, Theorem 5.3. In the convention required here, that result produces an invertible real matrix X, a real Jordan matrix J, and a canonical metric N_J satisfying

$$C = X^{-1} J X, \tag{3}$$

$$X M X^{\mathsf{T}} = N_J. \tag{4}$$

For a real-eigenvalue Jordan block of size n, the corresponding metric block is εP_n, where $\varepsilon \in \{1, -1\}$ and P_n is the reversal permutation matrix. A block for a non-real conjugate pair has real dimension $2m$ and an unsigned reversal metric; it has no sign characteristic.

The convention in (3)–(4) is obtained by applying the cited canonical-pair theorem to C^{T}, rather than directly to C. Indeed, (2) and $M^2 = I$ imply

$$CM = MC^{\mathsf{T}}. \tag{5}$$

The cited theorem then gives an invertible real matrix S, a real Jordan matrix J, and a canonical signed metric $P_{\varepsilon,J}$ such that

$$C^{\mathsf{T}} = S^{-1} J S, \tag{6}$$

$$M = S^{\mathsf{T}} P_{\varepsilon,J} S. \tag{7}$$

Set $X = S^{-\mathsf{T}}$. Transposing (6) and using (7) gives

$$C = S^{\mathsf{T}} J^{\mathsf{T}} S^{-\mathsf{T}}, \tag{8}$$

$$C = X^{-1} J^{\mathsf{T}} X, \tag{9}$$

$$XMX^{\mathsf{T}} = S^{-\mathsf{T}} M S^{-1}, \tag{10}$$

$$XMX^{\mathsf{T}} = P_{\varepsilon,J}. \tag{11}$$

It remains to restore the chosen upper-Jordan orientation. For every real Jordan block $J_k(\lambda)$, reversal satisfies

$$P_k J_k(\lambda)^{\mathsf{T}} P_k = J_k(\lambda), \tag{12}$$

$$P_k^2 = I. \tag{13}$$

Simultaneously reversing the chain and realification coordinates gives the corresponding identity for each non-real conjugate-pair block. Let R be the direct sum of these blockwise reversals. Replacing J^{T} by $RJ^{\mathsf{T}}R$ and X by RX preserves (9). It also preserves each metric block:

$$R(\varepsilon P_k)R^{\mathsf{T}} = \varepsilon P_k \tag{14}$$

for a real block, with the same identity for an unsigned conjugate-pair block. Thus $N_J := P_{\varepsilon,J}$ has the required orientation and sign convention. These inverse, transpose, and reversal operations are essential. An ordinary Euclidean singular value decomposition does not prove (3)–(4).

2 The four-dimensional inertia restriction

The reversal form P_n has inertia

$$\operatorname{inertia}(P_n) = \left(\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor \right). \tag{15}$$

Multiplication by a negative sign interchanges the two indices. An unsigned conjugate-pair block of real dimension $2m$ has inertia

$$\operatorname{inertia}(P_{2m}) = (m, m). \tag{16}$$

Total inertia $(1, 3)$ therefore excludes a real Jordan block of size four, a conjugate-pair chain of real dimension four, and two blocks of dimension two. The remaining dimension patterns are exactly those used in the proof of Theorem 3 under review:

1. four real one-dimensional blocks;
2. one two-dimensional non-real conjugate-pair block and two real one-dimensional blocks;
3. one real Jordan block of size two and two real one-dimensional blocks;

4. one real Jordan block of size three and one real one-dimensional block.

The phrase “orthogonal 2×2 block” in the second case denotes the real block for a non-real conjugate eigenvalue pair. It does not mean that the block is an orthogonal matrix for the Euclidean scalar product.

3 Formalized infrastructure

The formal development contains reversal matrices, their signed variants, and explicit congruences that expose the signatures of P_2, P_3, and P_4. It also defines real Jordan blocks and realifications of non-real conjugate-pair Jordan chains. The formalized block identities are

$$(\varepsilon P_n)J_n(a) = J_n(a)^\mathsf{T}(\varepsilon P_n), \tag{17}$$

$$P_{2m}J_m(a,\tau) = J_m(a,\tau)^\mathsf{T}P_{2m}. \tag{18}$$

The matrix M, the condition (2), and the four target block patterns are represented directly.[1]

4 The remaining obstruction

The formal library has no real Jordan canonical-form existence theorem that constructs J and a similarity matrix for an arbitrary real 4×4 matrix. More importantly, it has no formal canonical-pair theorem that simultaneously controls the similarity and the congruence class of the indefinite metric, including the sign characteristic. Sylvester's law for quadratic forms applies only after N_J and the congruence in (4) have been constructed; it cannot provide the missing simultaneous reduction.

No theorem with additional hypotheses is presented as the cited result. The existence statements (3) and (4), and the exhaustiveness of the four cases, remain unformalized. Closing the gap requires either a source-faithful real Jordan theorem followed by a finite-dimensional proof of the cited canonical-pair theorem, or a direct four-dimensional proof preserving the same block, sign, transpose, and inverse conventions. Before fixing the final formal signature, the exact existence clause in the cited edition must also be transcribed verbatim. The present translation of conventions is a high-confidence reconstruction from the statement context and proof, not a verbatim quotation.

[1] Formal declarations in `QICLean/Algebra/MinkowskiCanonicalPair.lean` include `Matrix.signedSipMatrix`, `Matrix.realJordanBlock`, `Matrix.realConjugatePairJordanBlock`, `Matrix.signedSipMatrix_mul_realJordanBlock`, `Matrix.complexPairSipMatrix_mul_realConjugatePairJordanBlock`, `Matrix.minkowskiMetric`, `Matrix.IsMinkowskiSelfAdjoint`, and `Matrix.GLRFourDimensionalPattern`.

5 Conclusion

This note records an open formalization dependency, not a correction to either cited source. The canonical blocks and all currently asserted algebraic identities are proved. The simultaneous similarity–congruence existence theorem remains the load-bearing missing result. Numerical orbit enumeration, Euclidean singular value decomposition, or a generic indefinite-SVD assumption would bypass rather than close the gap.