

Cross-Band Commutation in the Transport Estimate of the Area-Law Paper

(restriction resolved 2026-10-08)

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Verdict: scope-restriction (resolved).

1 The source hypothesis

Proposition 7.4 of Ref. [Ope26] (`06-transport.tex`, lines 377–434) concerns band metrics $A(P, Y, F) = (W_P^{-1}W_F^{-1}W_Y)^2$ on the symmetric subspace $\mathcal{S}_k = \text{Sym}^k \mathcal{V}$ (lines 251–258). Its commutation condition (lines 273–276) asks that every old or new metric of band g commute with every old or new metric of band $g' \neq g$. Since the metrics are operators on $\mathcal{S}_k$, this is commutation on $\mathcal{S}_k$. The paper uses this point when it applies the proposition: under the nesting condition of `07-comparators.tex` (lines 26–35), the different bands’ metric algebras commute only after complementary labels are identified on the symmetric space, using $W_{Q,k} = W_{Q^c,k}$ on $\mathcal{S}_k$ (`05-replicas.tex`, line 312).

2 The former formal restriction

The formal band metric is the operator $(W_P^{-1}W_F^{-1}W_Y)^2$ on the full tensor power $\mathcal{V}^{\otimes k}$. It commutes with the copy permutations, so it preserves $\mathcal{S}_k$ and restricts there to the source metric. The first formal cross-band hypothesis asked for commutation of these operators on all of $\mathcal{V}^{\otimes k}$, for every k .

Commutation on the full tensor power implies commutation on $\mathcal{S}_k$, but the converse fails. Central label functions of two subsystems Q, Q' commute on $\mathcal{V}^{\otimes k}$ when $Q \subseteq Q'$ or $Q \cap Q' = \emptyset$, but in general not when the two subsystems overlap without nesting. For three qubits a, b, c , $k = 3$, $t = 1/10$ and

the label weights of `05-replicas.tex` (display `replicas:w-definition`, padding dimension 8), a direct computation gives $\|[W_{ab}, W_{bc}]\|_2 \approx 3.9 \times 10^{-4}$ on $\mathcal{V}^{\otimes 3}$, while the commutator vanishes on $\mathcal{S}_3$, where $W_{ab} = W_c$ and $W_{bc} = W_a$.

In the comparator setting, $P_g = CXU_g$, $F_g = RV_g$, and for $g < g'$ the nesting condition gives $XU_gY_g \subseteq XU_{g'}$. Every pair of factors from the two bands is nested or disjoint, except $F_g = RV_g$ and $P_{g'} = CXU_{g'}$, which overlap in $V_g \cap U_{g'}$ without nesting. The remaining factors of each band commute with every factor of the other band, so the full-space commutation of A_g and $A_{g'}$ reduces to that of W_{RV_g} and $W_{CXU_{g'}}$, which fails in general. The first formal theorem therefore proved Proposition 7.4 under a commutation hypothesis that the source does not impose and that its multi-band application need not satisfy.

3 Resolution

Let Π be the projection onto $\mathcal{S}_k$. Each band metric A commutes with Π , and the mean trees now take as inputs the products over bands of the operators

$$\tilde{A} = \Pi A + (I - \Pi),$$

which equal A on $\mathcal{S}_k$ and the identity on its orthogonal complement.¹ The cross-band hypothesis is now the source condition: $AA'w = A'Aw$ for every $w \in \mathcal{S}_k$ and all metrics A, A' of different bands. The following facts make the transport argument go through unchanged.

1. $\tilde{A}$ is positive definite and commutes with the copy permutations.
2. $\tilde{A}\tilde{B} = \Pi AB + (I - \Pi)$, so $\tilde{A}$ and $\tilde{B}$ commute on $\mathcal{V}^{\otimes k}$ exactly when $\Pi AB = \Pi BA$, that is, when A and B commute on $\mathcal{S}_k$.
3. For a commuting projection, functional calculus acts block by block: $f(\Pi Y + (I - \Pi)Z) = \Pi f(Y) + (I - \Pi)f(Z)$. The proof interpolates f by one polynomial on the two finite spectra. Hence $\tilde{A}^{-1/2}\tilde{B}\tilde{A}^{-1/2} = \Pi A^{-1/2}B A^{-1/2} + (I - \Pi)$, and the relative coherent pin, whose lower bound $P_{\theta,k}$ satisfies $\Pi P_{\theta,k} \Pi = P_{\theta,k}$, transfers to the extended metrics.

¹Formal declarations: `TensorPower.ReplicaTransport.symBandMetric` and the restated `TensorPower.ReplicaTransport.TransportData.CrossBandCommute` in `QIClean/Representation/ReplicaTransport/Setup.lean`, with the properties below in `QIClean/Representation/ReplicaTransport/SymmetricMetric.lean`.

4. For X commuting with Π , the skew squares of $\tilde{A}$ and of A agree after multiplication by Π , so they have the same expectation in every density matrix on $\mathcal{S}_k$. The skew bound of the source therefore applies at a split leaf.

On $\mathcal{S}_k$ the roots, the filtered vectors and the transport states are those of the source, since every input preserves $\mathcal{S}_k$ and agrees there with the source's metric.

4 Verdict

Verdict: resolved. The transport estimate is formalized with the source's cross-band commutation condition on $\mathcal{S}_k$. Commutation on the full tensor power is no longer assumed.

References

- [Ope26] OpenAI. A two-dimensional area law from a global spectral gap, 2026. September 24, 2026; source revision adc7f1241b42e322a6451854ab7e4b4c146bf78a.