

The remaining stationarity step for the patch regulator

2026-10-07; resolved 2026-10-13

Verdict: scope-restriction (resolved).

1 Source argument

The September 24, 2026 manuscript *Polynomial PEPS approximation of gapped square-grid ground states* minimizes the norm of an ordered product of regional shifted powers. In `03-patches.tex`, lines 130–205, descending trace calculations give commuting regional minimizing densities x_j and output marginals ρ_j . The eigenvalue conditions `eq:patch-simplex-derivative`, `eq:patch-kkt`, and `eq:patch-clipped-eigenvalues` imply $\rho_j \leq x_j + bI$. The source is fixed at commit `adc7f1241b42e322a6451854ab7e4b4c146bf78a` of `openai/math`.

2 Proved operator consequences

The results in `QICLean/Analysis/PatchRegulator.lean` prove the following finite-dimensional implications. For commuting positive matrices x, ρ , $b > 0$, and $\rho \leq x + bI$, the actual regulator $T = b\rho(x + bI)^{-1}$ satisfies $0 \leq T \leq bI$ and $T \leq \rho$. For every orthogonal projection Q ,

$$\mathrm{tr} T \leq b \mathrm{rank} Q + \mathrm{tr}((I - Q)\rho). \quad (1)$$

This estimate does not require Q to commute with T or ρ . The source’s exponential rank and constant-tail inequalities imply $\mathrm{tr} T \leq 1/4$ at its stated starting radius. These are auxiliary implications with their hypotheses explicitly stated; they do not assert those hypotheses at every minimizer of the ordered patch objective.

3 Original continuation

The remaining work is to derive the commutation and derivative identities for the actual ordered objective by descending trace calculations, with the same normalized output marginals as in the source. Closed-simplex first-order conditions

can then imply the density order, and the regulator estimate can be applied without an additional hypothesis on minimizers. A separable scalar shifted-power objective is sufficient for the last-filter calculation; it must not be identified with the full ordered objective without this argument. Until the continuation is proved, the unconditional active-minimizer trace bound and Proposition 4.1 remain unformalized.

4 Resolution

The continuation is proved in `QIClean/Entropy/PatchStationarity.lean`. For an ordered list of factors with nested regions, positive exponents and density matrices x_j minimizing $\|L_m \cdots L_1 \Omega\|$ over the densities, every regional state ρ_j of the normalized output satisfies $[x_j, \rho_j] = 0$ and $\rho_j \leq x_j + bI$ (`Entropy.patch_stationarity`); consequently the regulator bounds hold at every minimizer (`Entropy.patch_shiftedRegulator_bounds`). No hypothesis beyond the source's is added.

The commutation is proved in decreasing order of j by the unitary variation $x_j \mapsto Ux_jU^*$ and the descending trace identity, as in the source. For the order, the source's coordinate conditions `eq:patch-simplex-derivative` and `eq:patch-kkt` are replaced by an equivalent operator argument. Vary x_j along the segment toward a density y commuting with x_j ; the one-sided first-order condition and the descending trace identity give $\text{Re tr}(Gy) \leq \text{Re tr}(Gx_j) =: \lambda$ with $G = \rho_j(x_j + bI)^{-1}$. Testing y against the rank-one projections of a common eigenbasis of G and x_j gives $G \leq \lambda I$, and $\lambda = \text{tr } \rho_j - b \text{tr } G \leq 1$. Hence $\rho_j = G(x_j + bI) \leq \lambda(x_j + bI) \leq x_j + bI$.

The density order, projection rank and complementary tail bounds used by `Matrix.PosSemidef.trace_shiftedRegulator_le_quarter` remain explicit hypotheses there; only the first two hypotheses of the regulator estimate (commutation and density order) are discharged here.