Nil-Matrix Product-Length Bound in the MPU Blocking Argument

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Verdict: local-correction (resolved).

1 The source assertion

Let A be a possibly nonunital associative subalgebra of the endomorphisms of a complex vector space of dimension D^2, and suppose that every element of A is nilpotent. The matrix-product-unitary blocking argument invokes the Nagata–Higman and Razmyslov results to choose a mixed-product length J' such that

$$J' < D^2, \tag{1}$$

$$A^{J'} = 0. \tag{2}$$

Here $A^{J'} = 0$ means that every ordered product of J' elements of A vanishes. This assertion appears in lines 405–426 of Ref. [CPGSV17].

The references cited there are Nagata, *Journal of the Mathematical Society of Japan* 4 (1952), 296–301; Higman, *Proceedings of the Cambridge Philosophical Society* 52 (1956), 1–4; and Razmyslov, *Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya* 38 (1974). Their nilpotency theorems and general quantitative bounds do not supply the strict linear estimate (1).

2 The corrected nil-matrix bound

Let V be an n-dimensional vector space over a field, and let $A \subseteq \mathrm{End}(V)$ be a possibly nonunital associative subalgebra whose elements are all nilpotent. The finite-dimensional nil-matrix theorem, also called Levitzki's theorem and obtainable from Engel's theorem by simultaneous strict triangularization, gives

$$A^n = 0. \tag{3}$$

Applying (3) to the D^2-dimensional transfer space gives

$$J' \leq D^2. \tag{4}$$

The weak inequality is sharp in general. The algebra of strictly upper-triangular $n \times n$ matrices contains a nonzero ordered product of $n-1$ matrix units. Therefore no general argument can replace (4) by the strict bound in (1).

3 Effect on the blocking estimate

The source has already obtained a first Jordan-block length J satisfying

$$J < D^2. \tag{5}$$

Combining (5) with (4) yields

$$JJ' < D^2 D^2, \tag{6}$$

$$JJ' < D^4. \tag{7}$$

The final strict blocking estimate is unchanged. Only the intermediate claim $J' < D^2$ must be replaced by $J' \leq D^2$.

The formal development proves (3) for arbitrary finite-dimensional vector spaces and transports it to square matrices.[1] It includes the zero-dimensional case and fixes the ordered-product convention. The strict-upper-triangular example explains why a strict bound is unavailable; no formalization of that example is claimed.

4 Classification

Local correction. Replacing the unsupported intermediate strict bound with the finite-dimensional nil-matrix bound preserves every hypothesis and conclusion required by the surrounding blocking argument. In particular, (7) retains the source's final strict estimate.

References

[CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.

[1] Formal declarations: `NonUnitalSubalgebra.list_prod_eq_zero_of_forall_isNilpotent` and `NonUnitalSubalgebra.matrix_list_prod_eq_zero_of_forall_isNilpotent` in `TNLean/Algebra/NilMatrixSubalgebra`.