

Scope of Ha's A_γ Witness for Choi-Type Maps

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Verdict: scope-restriction (wip).

1 Source theorem and witness

Let $d \geq 3$, let $1 \leq n \leq d - 2$, and let $D(X)$ denote the diagonal part of $X \in M_D(\mathbb{C})$. For the Weyl operator U_{k0} , Wolf's Example 3.1 defines the Choi-type map

$$T_C(X) = (d - n)D(X) - X + \sum_{k=1}^n D(U_{k0} X U_{k0}^\dagger). \quad (1)$$

The source identifies this formula as Equation (3.20) and states that T_C is positive and indecomposable [Wol12, Example 3.1]. The local source is `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 357–365. Wolf gives neither a proof nor a citation in that paragraph.

Ha proves the stronger atomicity assertion for the same family [Ha98, Theorem 2.1, pp. 593–595]. Fix $0 < \gamma < 1$. Ha takes all 3^d -th roots of unity, constructs 2-simple vectors $z_{r,i}$, and defines

$$A_r = \frac{1}{3^d} \sum_{i=1}^{3^d} |z_{r,i}\rangle\langle z_{r,i}|, \quad (2)$$

$$A_\gamma = \frac{1}{d} \sum_{r=1}^d A_r. \quad (3)$$

Let V_2 be the cone generated by projectors onto 2-simple vectors. The construction in (2)–(3) gives $A_\gamma \in V_2$. Ha derives the entries of A_γ from the root power sum and the exponent-collision identity in Equations (2.1)–(2.3) of Ref. [Ha98].

Write A^T for Ha's block transpose, namely transposition in the second tensor factor. With cyclic suffixes, Ha defines

$$u_i = \frac{\gamma}{\sqrt{d}} e_{i+1} \otimes e_i + \frac{1}{\sqrt{d}\gamma} e_i \otimes e_{i+1}, \quad (4)$$

$$v_i = \sqrt{\frac{d-1}{d}} (e_{i+1} \otimes e_i + e_i \otimes e_{i+1}), \quad (5)$$

$$\alpha_i = e_i \otimes e_i. \quad (6)$$

The source also defines vectors β_{1j} and β_{ij} for non-neighbouring pairs. Its calculation on p. 595 is

$$A_\gamma^T = \sum_{i=1}^d (|u_i\rangle\langle u_i| + |v_i\rangle\langle v_i| + |\alpha_i\rangle\langle \alpha_i|) + \sum_{j=3}^{d-1} |\beta_{1j}\rangle\langle \beta_{1j}| + \sum_{i=2}^{d-2} \sum_{j=i+2}^d |\beta_{ij}\rangle\langle \beta_{ij}|. \quad (7)$$

Every summand in (7) is generated by a 2-simple vector, so $A_\gamma^T \in V_2$. Ha finally computes

$$\langle A_\gamma, T_C \rangle = \gamma^2 - 1 < 0. \quad (8)$$

2 Formalized construction

Ha uses the one-based integers $m_k = \frac{3}{2}(3^{k-1} - 1)$ and the base vector $a_{i,1}$. The formalization uses zero-based indices, so the corresponding definitions are $m_k = \frac{3}{2}(3^k - 1)$ and $a_{i,0}$. The same convention defines the root family, the vectors $a_{i,k}$, c_r , $b_{r,i,j}$, and $z_{r,i}$, and the averages in (2)–(3). Cyclic coordinates lie in $\mathbb{Z}/d\mathbb{Z}$, consistently with Wolf’s Equation (2.24) [Wol12] and Ha’s rotation $Se_i = e_{i+1}$ [Ha98].

Fix r and i . Every column of $z_{r,i}$ except column r is a scalar multiple of $a_{i,0}$, while column r is $c_r \circ a_{i,r}$. Its coefficient matrix therefore has rank at most two. Averaging the associated rank-one projectors proves

$$A_\gamma \in V_2, \quad (9)$$

$$A_\gamma \geq 0. \quad (10)$$

The formalization also proves Ha’s root-average entry formula, Equation (2.3) of Ref. [Ha98]. It defines the vectors in (4)–(6) and the vectors $\beta_{i,j}$, with Ha’s two beta ranges translated to zero-based indices. Let $\mathcal{B}_d$ denote the union of those ranges, and let T_2 denote right partial transpose. The resulting identity and cone membership are

$$A_\gamma^{T_2} = \sum_{i=0}^{d-1} (|u_i\rangle\langle u_i| + |v_i\rangle\langle v_i| + |\alpha_i\rangle\langle \alpha_i|) + \sum_{(i,j) \in \mathcal{B}_d} |\beta_{i,j}\rangle\langle \beta_{i,j}|, \quad (11)$$

$$A_\gamma^{T_2} \in V_2. \quad (12)$$

For $d = 3$, the beta range is empty, as it is in the source. Thus the formalization proves both source-ordered V_2 decompositions on pp. 594–595 of Ref. [Ha98].¹

¹Formal declarations: `Matrix.haTwoSimpleVector_hasSchmidtRankLE_two`, `Matrix.haArgGamma_apply`, `Matrix.haAGamma_hasSchmidtNumberLE_two`, and `Matrix.haAGamma_posSemidef` in `QIClean/Channel/ChoiTypeMap/HaTwoSimpleWitness`; `Matrix.partialTransposeRight_haAGamma_eq_haBlockTransposeDecomposition` and `Matrix.partialTransposeRight_haAGamma_hasSchmidtNumberLE_two` in `QIClean/Channel/ChoiTypeMap/HaBlockTranspose`.

3 Factor swap, pairing, and remaining scope

Let σ swap the tensor factors, and define

$$J_d = \sum_{i,j} e_{ij} \otimes e_{ij}. \quad (13)$$

The Eom–Kye duality convention gives

$$\langle A, \Phi \rangle = \langle (\Phi \otimes \text{id})(A^\sigma), J_d \rangle. \quad (14)$$

See equations (12)–(13), Theorem 3.3, and the concluding shuffled pairing in Ref. [EK00, pp. 137–139]. Consequently, the state tested by the ampliation and its left partial transpose are

$$\rho = A_\gamma^\sigma, \quad (15)$$

$$\rho^{T_1} = (A_\gamma^{T_2})^\sigma. \quad (16)$$

The decompositions in (9) and (12) transport through the factor swap. The formalization therefore proves $\rho \in V_2$ and $\rho^{T_1} \in V_2$.

The formal development uses the normalized maximally entangled vector

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i e_i \otimes e_i, \quad (17)$$

$$J_d = d|\Omega\rangle\langle\Omega|. \quad (18)$$

It retains the positive factor d . For the full range $d \geq 3$, $1 \leq n \leq d - 2$, and $\gamma > 0$, the formalized identity is

$$\langle A_\gamma, T_C \rangle = d \langle \Omega | (T_C \otimes \text{id})(A_\gamma^\sigma) | \Omega \rangle, \quad (19)$$

$$\langle A_\gamma, T_C \rangle = \gamma^2 - 1. \quad (20)$$

Thus $0 < \gamma < 1$ gives the strict negativity in (8).²

This proves the source algebra of the witness. Positivity throughout the source range is also formalized, and the former middle-range gap is resolved in GitHub issue #19.³ A theorem combining positivity with the negative witness to conclude indecomposability remains to be stated; GitHub issue #18 tracks this final step. The present formalization does not assert atomicity.

²The factor swap, partial-transpose transport, pairing identity, cyclic-coordinate calculation, and normalized quadratic form are in `QICLean/Channel/ChoiTypeMap/HaNegativePairing`; the principal declarations are `Matrix.partialTransposeLeft_tensorFactorSwap_haAGamma_hasSchmidtNumberLE_two`, `Matrix.eomKyePairing_eq_omegaVec_quadraticForm_factorSwap`, `Matrix.eomKyePairing_haAGamma_choiTypeMap`, and `Matrix.choiTypeMapFin_haAGamma_omegaVec_quadraticForm_re_neg`.

³Formal declaration: `Matrix.choiTypeMap_isPositiveMap`.

4 Classification

Scope restriction. The formalization covers the exact root-average construction, its entry formula, both V_2 decompositions on pp. 594–595 of Ref. [Ha98], their tensor-factor transport, the Eom–Kye orientation, the normalization factor d , and the value $\gamma^2 - 1$ for every $d \geq 3$ and $1 \leq n \leq d - 2$. It also proves positivity in this range. The final indecomposability theorem remains open. No atomicity conclusion or alternative Choi-matrix or partial-transpose ansatz is asserted.

References

- [EK00] Myoung-Hoe Eom and Seung-Hyeok Kye. Duality for positive linear maps in matrix algebras. *Mathematica Scandinavica*, 86(1):130–142, 2000.
- [Ha98] Kil-Chan Ha. Atomic positive linear maps in matrix algebras. *Publications of the Research Institute for Mathematical Sciences*, 34(6):591–599, 1998.
- [Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.