Zero correlation length uses the physical-trace transfer in Cirac–Perez-Garcia–Schuch–Verstraete 2017

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Verdict: local-correction (resolved).

1 Source statement

For mixed states, Definition 4.2 of Ref. [CPGSV16], with local label `DefinitionZCL` in `Papers/1606.00608/MPDO-22-12-17-2.tex:735--739`, defines a tensor M generating an MPDO to have *zero correlation length* (ZCL) through the graphical equation in `MPDO_ZCL.png`. Let M^{ij} denote the bond matrix with ket index i and bra index j. The diagram closes the ket and bra physical legs of one MPDO tensor, so it represents the physical-trace transfer

$$\mathcal{T}_M := \sum_i M^{ii}, \tag{1}$$

not the doubled-index completely positive map on bond matrices. The source calls this condition "the natural extension of the pure state case of Theorem `TheoremZCLPure`" and states that correlation functions computed from an MPDO with ZCL are length independent [CPGSV16, lines 735–739].

The proof of Theorem `TheoremZCLPure` in Ref. [CPGSV16], at `Papers/1606.00608/MPDO-22-12-17-2.tex:1248`, states that pure-state ZCL is equivalent to

$$\mathbb{E}^2 = \mathbb{E} \tag{2}$$

for a tensor A in canonical form. The renormalization step is

$$\mathcal{E} \longmapsto \mathcal{E}^2, \tag{3}$$

as stated at `Papers/1606.00608/MPDO-22-12-17-2.tex:1209`. A fixed point therefore satisfies

$$\mathcal{E} = \mathcal{E}^2. \tag{4}$$

Canonical form normalizes the leading eigenvalue of the transfer operator to one. Equation (2) then states that $\mathbb{E}$ projects onto its peripheral eigenvalue-one subspace.

2 The gap

Define the doubled-index completely positive transfer map by

$$\mathcal{E}_M(X) = \sum_{i,j} M^{ij} X (M^{ij})^*. \tag{5}$$

The older Lean definition `MPOTensor.IsZCL` instead requires

$$\mathcal{E}_M \circ \mathcal{E}_M = \mathcal{E}_M. \tag{6}$$

This is literal idempotence of the doubled-index completely positive transfer map, not the transfer drawn in Definition 4.2 of Ref. [CPGSV16]. The source first contracts the two physical legs and then squares $\mathcal{T}_M$ from Eq. (1). Thus the older predicate is not merely an unnormalized version of the source condition; it concerns a different transfer object. Even up to scalar normalization, idempotence of $\mathcal{E}_M$ classifies tensors through the doubled-index MPS transfer rather than through the MPDO ZCL diagram.

Classification: resolved definition drift. The older formal predicate records doubled-index CP-transfer idempotence, whereas the source definition uses the physical-trace transfer. The source definition is now formalized by `MPOTensor.IsPhysicalTraceIdempotent`, whose equation characterization is

$$\mathcal{T}_M^2 = \mathcal{T}_M. \tag{7}$$

The predicate `MPOTensor.IsSourceZCL` records a separate, scale-invariant up-to-scalar relation. It is distinct from both the paper's literal diagram and the doubled-index predicate.

The development witnesses this deviation through `MPOTensor.exists_isLocalPurificationRFP_not_isZCL`. Take $d = d_K = 2$, $D' = D = 1$, and

$$A = \left[\tfrac{1}{\sqrt{2}}, 0, 0, \tfrac{1}{\sqrt{2}}\right]. \tag{8}$$

This is an RFP MPS tensor with

$$E_A = \mathrm{id}\,. \tag{9}$$

The induced MPDO tensor has

$$M^{00} = M^{11} = \tfrac{1}{2}, \tag{10}$$

$$M^{01} = M^{10} = 0. \tag{11}$$

Its length-N periodic operator is

$$\rho^{(N)}(M) = \left(\tfrac{1}{2}\right)^N I. \tag{12}$$

This normalized maximally mixed state has length-independent correlations and therefore satisfies ZCL in the source's sense. Its physical-trace transfer is

$$\begin{aligned} \mathcal{T}_M &= M^{00} + M^{11} && (13) \\ &= 1, && (14) \end{aligned}$$

so the graphical idempotence holds. By contrast, its doubled-index CP transfer satisfies

$$\begin{aligned} \mathcal{E}_M &= \tfrac{1}{2}\,\mathrm{id}, && (15) \\ \mathcal{E}_M \circ \mathcal{E}_M &= \tfrac{1}{4}\,\mathrm{id} && (16) \\ &\neq \mathcal{E}_M. && (17) \end{aligned}$$

Thus literal `MPOTensor.IsZCL` fails. The witness exposes the wrong transfer object, not merely a missing scalar normalization.

3 Consequence and elimination plan

The source implication PRFP $\Rightarrow$ ZCL at `Papers/1606.00608/MPDO-22-12-17-2.tex:777` in Ref. [CPGSV16] cannot be represented by `MPOTensor.IsZCL`: the source implication controls $\mathcal{T}_M$, whereas that predicate controls $\mathcal{E}_M$. After this definition drift is repaired, the printed implication to literal physical-trace idempotence is itself refuted by `MPOTensor.exists_isPRFP_isMPDO_physTraceTransfer_ne_zero_not_isPhysicalTraceIdempotent`. Its witness satisfies global PRFP and MPDO positivity with

$$\begin{aligned} \mathcal{T}_M &\neq 0, && (18) \\ \mathcal{T}_M^2 &\neq \mathcal{T}_M. && (19) \end{aligned}$$

The stronger failure of the scale-invariant relation remains a separate project result.

Two normalization conventions remain useful for corrected restricted theorems:

1. The source ZCL transfer uses $\mathcal{T}_M = \sum_i M^{ii}$ and allows scalar normalization at the level of the generated state family. Its nondegenerate up-to-scalar condition is

   $$\begin{aligned} &\mathcal{T}_M \neq 0, && (20) \\ &\mathcal{T}_M^2 = \lambda \mathcal{T}_M \qquad \text{for some } \lambda > 0. && (21) \end{aligned}$$

   The physical-trace-normalized representative is the case $\lambda = 1$.

2. The canonical physical-trace hypothesis retains exact idempotence in Eq. (7) only for source-labelled theorems whose hypotheses include the required canonical normalization. The doubled-index map $\mathcal{E}_M$ remains a separate completely positive transfer used in the formal fusion and algebra statements, not the source's ZCL transfer.

The source definition and the scale-invariant relation are now both formalized. The declaration `MPOTensor.IsPhysicalTraceIdempotent` records the literal fixed-tensor diagram, `MPOTensor.IsSourceZCL` records the nonzero up-to-scalar relation, and `MPOTensor.IsZCL` retains the doubled-index CP-transfer condition. These predicates must remain distinct.

For the renormalization condition in Definition 4.1 of Ref. [CPGSV16], Proposition `propsimple` is now proved at the explicit normalization boundary: every positive-length ring is positive semidefinite and has nonzero trace. The primary source-facing declarations are `MPOTensor.IsPhysicalTraceIdempotent` for Definition 4.2 and `MPOTensor.isPhysicalTraceIdempotent_of_isRFPViaTS` for its RFP implication, together with `MPOTensor.isSAL_of_isRFPViaTS_of_trace_ne_zero` for SAL. The conjunction `MPOTensor.isSourceZCL_and_isSAL_of_isRFPViaTS_of_trace_ne_zero` has the broader scale-invariant conclusion and is a convenience theorem, not the paper-facing statement.

The horizontal source-facing specialization uses `MPOTensor.physTraceTransfer_sq_of_isRFPViaTS` together with `MPOTensor.isSAL_of_isRFPViaTS`. The corresponding `MPOTensor.IsSourceZCL` conjunction is again only a convenience result. Positivity alone admits the zero tensor, so the trace condition cannot be omitted. The printed global PRFP equivalence is refuted; the local-purification equivalences remain valid only under their stated tensor-level restriction.

In Theorem 4.9 of Ref. [CPGSV16], the literal implication (iv)$\Rightarrow$(v) is refuted by the unequal repeated-copy example described below. The source route (ii)$\Rightarrow$(v) remains unproved after the ZCL-derived common-weight reduction.

Amendment for the fixed-tensor implication in Theorem 4.9. The first convention is scale invariant at the level of the generated state family, but it cannot be substituted unchanged into the implication to the exact trace-preserving maps of Definition 4.1 in Ref. [CPGSV16]. The four-sector audit in `docs/paper-gaps/cpgsv17_pf_rank_one.tex` exhibits the distinction. Its raw tensor $\mathcal{K}$ satisfies

$$\mathcal{T}_{\mathcal{K}}^2 = \mathcal{T}_{\mathcal{K}}, \tag{22}$$

and has explicit renormalization maps, but its singleton BNT coefficient is strictly subunit. The globally unit-weight representative

$$A = \frac{4}{\sqrt{5}}\mathcal{K} \tag{23}$$

satisfies `MPOTensor.IsSourceZCL` only up to scalar:

$$\begin{aligned} \mathcal{T}_A^2 &= \frac{4}{\sqrt{5}}\mathcal{T}_A \qquad (24)\\ &\neq \mathcal{T}_A. \qquad (25) \end{aligned}$$

Its two-site blocking consequently fails Definition 4.1 of Ref. [CPGSV16]. The scalar absorption at source line 1660 is performed only after literal ZCL has been assumed and does not transport literal ZCL or the renormalization maps across rescaling. Thus the first convention is the scale-invariant interpretation used by TNLean, whereas the paper's fixed representative retains a separate literal diagrammatic condition. Neither four-sector representative settles the exact (ii)$\Rightarrow$(v) route.

A separate scalar BNT presentation with raw copy weights 1 and 1/2 satisfies the global normalization and every clause of condition (iv), while its two-site blocking fails Definition 4.1. The kernel-checked counterexample is `MPOTensor.CaseIIAbsorptionCounterexample.printed_theorem49_iv_to_v_is_false`; it refutes the literal implication (iv)$\Rightarrow$(v).

4 The scale-invariant repaired interpretation and its consequences

The repaired interpretation used in this development is the first convention. In terms of the physical-trace transfer from Eq. (1), the predicate `MPOTensor.IsSourceZCL` requires

$$\mathcal{T}_M \neq 0, \tag{26}$$

$$\mathcal{T}_M^2 = \lambda \mathcal{T}_M \qquad \text{for some } \lambda \in \mathbb{R} \text{ with } \lambda > 0. \tag{27}$$

The nonzero clause excludes the degenerate zero tensor, which would otherwise satisfy the scalar relation vacuously for every λ. No spectral-radius apparatus is required. If $\mathcal{T}_M^2 = \lambda \mathcal{T}_M$ with $\lambda > 0$ and $\mathcal{T}_M \neq 0$, then the minimal polynomial of $\mathcal{T}_M$ divides

$$x(x - \lambda). \tag{28}$$

Its eigenvalues therefore lie in

$$\{0, \lambda\}, \tag{29}$$

and λ is the leading eigenvalue. Literal idempotence is exactly the case $\lambda = 1$, corresponding to the physical-trace-normalized representative. The maximally mixed witness has $\mathcal{T}_M = 1$ and is therefore classified as ZCL by the source transfer, whereas the doubled-index condition excludes it.

The correction is to ZCL, not to the doubled-index ZCL predicate. The doubled-index transfer map $\mathcal{E}_M$ remains the object used by `MPOTensor.IsZCL` and by the transfer-map formulation of the fusion and algebra statements. It must not be identified with the source ZCL diagram. Literal physical-trace idempotence implies normalized ZCL only together with $\mathcal{T}_M \neq 0$, or an equivalent generated-MPDO nondegeneracy assumption. Without that clause, the zero transfer is idempotent but does not satisfy Eqs. (26) and (27). Every source-facing description that calls the doubled-index literal notion "definitionally zero correlation length" must be weakened to a separate bridge with the nonzero hypothesis stated explicitly.

Definition 4.1 gives the forward physical-trace equation. The theorem `MPOTensor.isPhysicalTraceIdempotent_of_isRFPViaTS`, using the physical-trace-square calculation, formalizes the ZCL part of Proposition `propsimple` in Appendix C of Ref. [CPGSV16], at source lines 1333–1340. That proposition proves the first implication of Theorem 4.9 by establishing both ZCL and SAL. At the explicit density-family boundary, `MPOTensor.isSourceZCL_and_isSAL_of_isRFPViaTS_of_trace_ne_zero` combines the physical-trace theorem with positive-length nonvanishing and `MPOTensor.isSAL_of_isRFPViaTS_of_trace_ne_zero`. The horizontal declarations are corollaries obtained by deriving that nonvanishing.

For every virtual matrix X, trace preservation of the one-to-two map gives

$$
\begin{aligned}
\operatorname{tr}(\mathcal{T}_M^2 X) &= \operatorname{tr}(M_2(X)) && (30)\\
&= \operatorname{tr}(M_1(X)) && (31)\\
&= \operatorname{tr}(\mathcal{T}_M X). && (32)
\end{aligned}
$$

Nondegeneracy of the trace pairing yields Eq. (7). Together with $\mathcal{T}_M \neq 0$, this implies `MPOTensor.IsSourceZCL`. The explicit nonzero hypothesis is necessary because the bare predicate `IsRFPViaTS` omits the source's standing canonical-form and normalized-MPDO assumptions.

Effect on the purification witness. The diagonal purification in Eq. (8) has $\mathcal{T}_M = 1$ and therefore already satisfies the source ZCL equation. The earlier claim that this witness fails zero correlation length resulted from using $\mathcal{E}_M = \frac{1}{2}\,\mathrm{id}$ in place of the physical-trace transfer. Under the scale-invariant repaired interpretation, it is a positive example showing that the up-to-scalar condition captures the maximally mixed product state. This conclusion does not identify that condition with the paper's literal fixed-representative diagram.

Historical resolution of the doubled-index coercion. Commit `43fb8749` introduced the former blocked-RFP construction, whose proof treated doubled-index idempotence as the zero-correlation-length clause of Theorem 4.9 in Ref. [CPGSV16]. Commit `74fb83dd` marked that dependence as unfaithful. Commit `32482c47` removed the coercion and the paper-facing theorem chain, and commit `fc135934` removed the remaining obsolete module. No such coercion route remains in the development.

The present formalization separates the three transfer notions. The literal Definition 4.2 conclusion

$$
\mathcal{T}_M^2 = \mathcal{T}_M \tag{33}
$$

follows directly from Definition 4.1 in `MPOTensor.physTraceTransfer_sq_of_isRFPViaTS`. The broader scale-invariant relation is recorded by `MPOTensor.IsSourceZCL`, while `MPOTensor.IsZCL` concerns a different doubled-index transfer and is not used as the source's ZCL clause. The doubled-index results in `FusionIsometries.lean`, `AlgebraStructure.lean`, and `CommutingFormBridge.lean` survive only as separately scoped conditional helpers. None is a relocated proof of Theorem 4.9.

The normal Case-I structural conclusion is proved by `MPOTensor.exists_physicalSectorFactorization_rank_one_coefficients_of_isSAL_of_literal_ZCL`. Given one such factorization, `MPOTensor.PhysicalSectorFactorization.NeighboringTraceFactorization.blockTwo_isRFPViaTS` composes the two physical channels of Proposition C.7 in Ref. [CPGSV16] twice, as observed at source lines 1821–1825, and transports them back to the original physical coordinates.

The remaining obligations belong to the stronger route from condition (ii) of Theorem 4.9 in Ref. [CPGSV16]. First, the Case-II argument needs a replacement for the false claim that coefficient absorption preserves normality. This is the *missing statement* in the (ii)⇒(iv) node and is analyzed in `docs/paper-gaps/cpgsv17_pf_rank_one.tex`.

Second, the Case-II argument at lines 1646–1665 of Ref. [CPGSV16] uses literal ZCL to make repeated-copy weights independent of the copy label and absorb their common value. Lemma C.10 is linked exactly by `MPOTensor.weight_copy_independent_of_isPhysicalTraceIdempotent`, with its simple-canonical existential form supplied by `MPOTensor.IsSimpleCanonicalForm.exists_weight_copy_independent_of_isPhysicalTraceIdempotent`. Both results follow directly from the literal weighted-block equation. The older `IsSourceZCL` declarations remain separate scale-invariant generalizations and are no longer used as source nodes. After this normalization, the Proposition C.7 channel pairs must still be combined, using the mutually orthogonal outer physical supports, into a single pair of trace-preserving completely positive maps. This is the *missing statement* in the (ii)⇒(v) node.

By contrast, the raw implication (iv)⇒(v) is not a remaining assembly problem. It is a refuted *unfaithful statement*, and treating the one-factorization result as though outer-sector assembly would prove it is *proof-path drift*. The hidden use of ZCL through `lemmus` in the proof of `prop3to4` in Ref. [CPGSV16] is a separate instance of the same proof-path drift. The former doubled-index theorem was removed rather than renamed. None of the surviving gaps is a doubled-index bridge or a change of terminology.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.