The CPSV16 Unit-Weight Convention and Renormalization-Fixed-Point Scale

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Verdict: false-source (resolved).

1 Two scale conventions in the source

The canonical-form construction of Cirac–Pérez-García–Schuch–Verstraete first writes a tensor as

$$A^i = \bigoplus_k \mu_k A_k^i, \tag{1}$$

where every A_k is normal and its transfer map has spectral radius one [CPGSV16, lines 214–246]. Since the states are not normalized there, line 246 adopts the standard convention

$$|\mu_k| \leq 1 \qquad \text{for every } k, \tag{2}$$

$$|\mu_{k_0}| = 1 \qquad \text{for some } k_0. \tag{3}$$

The formal BNT predicate records Eq. (3) through `MPSTensor.IsBNTCanonicalForm.weight_unit_exists`. This condition gives one global unit-weight witness, not one witness in every sector. The latter distinction is discussed separately in `docs/paper-gaps/cpsv16_global_vs_persector_unit_witness.tex`.

Definition 4.1 of Ref. [CPGSV16] fixes the tensor scale differently. For a tensor M already in canonical form and generating MPDOs, it requires trace-preserving completely positive maps relating the one-site and two-site physical closures [CPGSV16, lines 645–660]. Define the physical-trace transfer by

$$\mathcal{T}_M = \sum_i M^{ii}. \tag{4}$$

Trace preservation gives

$$\mathcal{T}_M^2 = \mathcal{T}_M, \tag{5}$$

as calculated at lines 1333–1340 of Ref. [CPGSV16]. The source-facing formal statement is `MPOTensor.isPhysicalTraceIdempotent_of_isRFPViaTS`; its proof uses `MPOTensor.physTraceTransfer_sq_of_isRFPViaTS`.

These requirements concern two transfer objects with different scaling laws. Define the doubled-index MPS transfer map by

$$\mathcal{E}_M(X) = \sum_i M^i X (M^i)^\dagger. \tag{6}$$

For a scalar c,

$$\mathcal{E}_{cM} = |c|^2 \mathcal{E}_M. \tag{7}$$

Linearity of the physical-trace transfer gives

$$\mathcal{T}_{cM} = c\mathcal{T}_M. \tag{8}$$

Equations (2) and (3) normalize the first object by choosing normal BNT representatives and placing the overall scale in their coefficients. Equation (5) constrains the second object at the scale of the displayed tensor. The related up-to-scalar treatment of source zero correlation length is recorded in `docs/paper-gaps/cpsv16_zcl_canonical_form_normalization.tex`.

2 Fixed-representative scale pinning

By Eq. (8), replacing M by cM replaces $\mathcal{T}_M$ by $c\mathcal{T}_M$. Suppose that $\mathcal{T}_M^2 = \mathcal{T}_M \neq 0$ and that the nonzero rescaling cM satisfies the same literal idempotence equation. Then

$$\begin{aligned} c^2\mathcal{T}_M &= (c\mathcal{T}_M)^2 \qquad (9)\\ &= c\mathcal{T}_M, \qquad (10) \end{aligned}$$

so $c = 1$. Thus Definition 4.1 of Ref. [CPGSV16] is not invariant under arbitrary nonzero rescaling of a fixed representative.

By contrast, the simplicity definition at lines 815–822 of Ref. [CPGSV16] is phrased through the elements of a BNT: the tensor is simple when none of those elements has nilpotent ket-against-bra contraction. Multiplying one BNT element by a nonzero scalar does not change nilpotency. This observation concerns only the nilpotency clause; it does not imply that arbitrary complex rescaling preserves the MPDO condition or either full simplicity predicate.

Let $\rho_N(M)$ denote the closed length-N MPO. Its exact scaling law is

$$\rho_N(cM) = c^N \rho_N(M), \tag{11}$$

not $|c|^{2N}\rho_N(M)$. Consequently, strictly positive real rescaling rM preserves positivity in both directions. The formal theorems `MPOTensor.isMPDO_smul_ofReal_iff` and `MPOTensor.isSimple_smul_ofReal_iff` establish the corresponding invariances for the MPDO and simplicity predicates. They do not assert invariance under arbitrary nonzero complex scalars, and they do not imply that the rescaled tensor satisfies the same literal fixed-scale RFP equations of Definition 4.1 of Ref. [CPGSV16].

The declaration `MPOTensor.IsSimple` formalizes lines 217–246 and Definition 4.7 of Ref. [CPGSV16]. It asks for a positive blocking length and a BNT sector presentation whose representative physical-trace transfers are all non-nilpotent. Every representative has a positive number of copies, and every copy has nonzero weight, following the nonzero-coefficient convention discussed in `docs/paper-gaps/cpsv16_bnt_uniqueness_zero_coefficient.tex`. The theorem `MPOTensor.IsSimple.exists_mpo_ne_zero` derives a positive length N such that

$$\rho_N(M) \neq 0, \tag{12}$$

from the presentation; this is not an additional defining assumption. Individual lengths may still vanish.

The predicate does not impose the line-246 unit-weight convention. Requiring a unit-weight witness for the blocked tensor would be a hyper-literal reading excluded by the source's own conventions. The dimer R considered below is simple in the source's sense because its physical-trace transfer is non-nilpotent, and it carries canonical-form weight $\sqrt{337/512}$. It would therefore not be simple under a unit-weight reading. Example 4.12 of Ref. [CPGSV16] is not a second such witness: its second BNT basis element has vanishing one-by-one physical-trace transfer, so it is not simple under either reading, and its canonical-form weight $1/\sqrt{2}$ is beside the point. See `docs/paper-gaps/cpsv16_simple_tensor_nilpotency.tex`. The library therefore treats line 246 as an available source normalization, not as a clause of Definition 4.7. The predicates that once separated these readings, and the counterexample module that separated them, were removed; see `docs/audits/2026-08-24_degenerate_readings_wave_2.md`.

By contrast, `MPOTensor.IsSimpleCanonicalForm` is a predicate on one displayed canonical-form representative and supplies the Appendix C.2 hypothesis of Ref. [CPGSV16] for an already-blocked tensor. It is not a quotient by nonzero scalar rescaling, and no equivalence with `MPOTensor.IsSimple` is asserted.

3 The parity tensor in Example 4.12

Example 4.12 of Ref. [CPGSV16] prints the two physical letters $M^{00} = I$ and $M^{11} = \sigma_z$. Its physical-trace transfer satisfies

$$\mathcal{T}_M^2 = 2\mathcal{T}_M, \tag{13}$$

so the two trace-preserving channel equations cannot hold for the printed representative. Define the density-normalized representative by

$$\widehat{M} = \tfrac{1}{2}M. \tag{14}$$

Its physical-trace transfer is idempotent. Explicit parity coarse-graining and refinement channels prove `MPOTensor.CPSVExample412NormalizedRFP.Mhat_isRFPViaTS` without any additional hypothesis; the complete calculation appears in `docs/paper-gaps/cpsv16_example_4_12_normalization.tex`.

The doubled-index MPS has two bond-one sectors, with physical vectors $(1,0,0,1)$ and $(1,0,0,-1)$. Each vector has squared norm 2, so division by $\sqrt{2}$ gives its normal representative. Consequently, the two horizontal normal-block weights of $\widehat{M}$ are both $1/\sqrt{2}$. The scale selected by the channel equations therefore does not supply the global unit-weight witness in Eq. (3). The formal theorem deliberately records the bare channel predicate and does not claim that $\widehat{M}$ satisfies every standing canonical-form convention in Definition 4.1 of Ref. [CPGSV16].

4 The displayed dimer representative

The project tensor R gives a concrete instance of this tension. Its doubled-index MPS transfer map satisfies

$$\mathcal{E}_R^2 = \frac{337}{512}\mathcal{E}_R. \tag{15}$$

Consequently, its displayed one-block CPSV canonical form is

$$R^{\bullet\bullet} = \mu \widehat{R}, \tag{16}$$

$$\mu = \sqrt{\frac{337}{512}}, \tag{17}$$

where the transfer map of the retained block $\widehat{R}$ is idempotent and therefore has spectral radius one. This presentation has $0 < \mu < 1$, so it does not supply the global unit-weight witness in Eq. (3).

The physical-trace transfer of the displayed tensor R is a nonzero idempotent; its $(0,0)$ entry is $1/2$. The retained block satisfies

$$\mathcal{T}_{\widehat{R}} = \mu^{-1}\mathcal{T}_R. \tag{18}$$

Since $\mu \neq 1$, the scale-pinning calculation in Eqs. (9) and (10) shows that this normalized representative does not satisfy the same literal physical-trace idempotence equation as R.

The obstruction is stronger than this displayed-presentation mismatch. For every positive blocking length L, the doubled-index transfer map of the blocked tensor satisfies

$$\mathcal{E}_{R^{[L]}}^2 = \left(\frac{337}{512}\right)^L \mathcal{E}_{R^{[L]}}. \tag{19}$$

Suppose that $R^{[L]}$ admitted a normalized horizontal BNT witness. The gauge to that witness transports Eq. (19). The global unit-weight condition from line 246 of Ref. [CPGSV16] supplies a copy whose weight has modulus one. Restricting the transported equation to that copy removes the weight. Its normal representative is left canonical, so its transfer map is trace preserving. Taking the trace of the quasi-idempotence equation at the identity forces

$$\left(\frac{337}{512}\right)^L = 1. \tag{20}$$

This contradicts

$$0 < \left(\frac{337}{512}\right)^L < 1. \tag{21}$$

Hence no positive blocking of R admits a normalized horizontal canonical form: no rescaling of R reconciles the line-246 normalization with the scale pinned by Definition 4.1 of Ref. [CPGSV16].

This argument is not a nilpotency statement and does not bear on Definition 4.7 of Ref. [CPGSV16]. The retained-block theorem proves that the displayed canonical presentation has non-nilpotent physical-trace transfer. Together with positivity and the one-site BNT refinement, it witnesses `MPOTensor.IsSimple` R. The theorem `R_isSimple` in the namespace `MPOTensor.RescalingStableLengthDependentRFP` proves this assertion. The source quantifier is existential and is witnessed by $L = 1$; no assertion is made about every positive blocking. Reading line 246 into Definition 4.7 would instead make R non-simple, which is the reading rejected here.

5 Coefficient absorption with all Appendix C.2 conditions

The two-sector coefficient-absorption example realizes the normalization tension without dropping the conditions used in the simple Case-II passage [CPGSV16, lines 1628–1665 and 1740–1782]. Its nonzero physical slices are

$$K^{00} = K^{11} = \mathrm{diag}(1/2, 0), \tag{22}$$

$$K^{22} = \mathrm{diag}(0, 1). \tag{23}$$

For $N > 0$,

$$\langle \sigma | \rho_N(K) | \tau \rangle = \delta_{\sigma,\tau} \left(2^{-N} \mathbf{1}_{\{\sigma_k \in \{0,1\}\ \forall k\}} + \mathbf{1}_{\{\sigma_k = 2\ \forall k\}} \right), \tag{24}$$

$$\mathrm{tr}\, \rho_N(K) = 2 > 0. \tag{25}$$

The tensor therefore generates positive semidefinite periodic operators with nonzero normalization.

Define

$$
\begin{aligned}
P &= \operatorname{diag}(1, 1, 0), && (26)\\
Q &= I - P, && (27)\\
\omega_P &= \tfrac{1}{4}(P \otimes P), && (28)\\
\omega_Q &= Q \otimes Q. && (29)
\end{aligned}
$$

The one-site and two-site physical closures are

$$
\begin{aligned}
M_1(X) &= \tfrac{X_{00}}{2} P + X_{11} Q, && (30)\\
M_2(X) &= X_{00}\omega_P + X_{11}\omega_Q. && (31)
\end{aligned}
$$

Define the partial-trace and measure-and-prepare maps by

$$
\begin{aligned}
\mathcal{S}(Y) &= \operatorname{tr}_2(Y), && (32)\\
\mathcal{R}(Y) &= \operatorname{tr}(PYP)\omega_P + \operatorname{tr}(QYQ)\omega_Q. && (33)
\end{aligned}
$$

These maps are trace preserving and completely positive, and they satisfy

$$
\begin{aligned}
\mathcal{S}[M_2(X)] &= M_1(X), && (34)\\
\mathcal{R}[M_1(X)] &= M_2(X). && (35)
\end{aligned}
$$

Equations (34) and (35) are the explicit Definition 4.1 equations from lines 638–660 of Ref. [CPGSV16]. They give SAL, while the slice sum gives the literal Definition 4.2 identity

$$
\begin{aligned}
\mathcal{T}_K &= I_2, && (36)\\
\mathcal{T}_K^2 &= \mathcal{T}_K. && (37)
\end{aligned}
$$

The displayed representatives are normal and biCF, the weights $(1/\sqrt{2}, 1)$ satisfy the global unit-weight convention, and the ambient tensor has normalized fixed-representative simple canonical form. Nevertheless,

$$
\begin{aligned}
\rho(\mathcal{E}_{\mu_0 A_0}) &= \frac{1}{2}, && (38)\\
\mu_0 A_0 &\text{ is not normal.} && (39)
\end{aligned}
$$

The component results and their conjunction are in `TNLean/MPS/MPDO/CaseIIAbsorptionCounterexample.lean`, culminating in `MPOTensor.CaseIIAbsorptionCounterexample.ambient_isSAL_isSimpleCanonicalForm_and_firstAbsorbed_not_isNormalTensor`. This repairs the previously missing full-condition counterexample statement and classifies the printed absorption inference as *proof-path drift*. It does not decide the claimed all-sector factorization.

The non-Cartesian witness in `TNLean/MPS/MPDO/NonCartesianActiveSectorCounterexample.lean` refutes the implication from injectivity, SAL, literal ZCL, and a scalar relation to a normal tensor. Let μ be its singleton coefficient and let r be the Perron value of its unnormalized transfer, as defined in `docs/paper-gaps/cpgsv17_pf_rank_one.tex`. They satisfy

$$|\mu|^2 = r, \tag{40}$$

$$0 < |\mu| = \sqrt{r} < 1. \tag{41}$$

The strict bound in Eq. (41) is formalized by `MPOTensor.NonCartesianActiveSectorCandidate.transferMap_posDef_eigenvalue_lt_one`. Thus the unit clause of the line-246 convention has no witness. Rescaling to unit weight destroys literal ZCL. Accordingly, GitHub issue #6775 remains open at the ambient source-context boundary.

This open issue is also the remaining input to the source's channel construction. Proposition `3to5` of Ref. [CPGSV16] constructs the representative maps from the neighboring-trace factorization, and Proposition `prop2to5` combines them after projecting onto the orthogonal outer subspaces. The latter combination is formalized in GitHub issue #6632. The direct sector-mixing alternative formerly tracked in GitHub issue #6793 is not a separate source step.

6 Unequal raw copies refute Theorem 4.9(iv)⇒(v)

The scale mismatch is decisive even when the source's global unit-weight convention is satisfied. Let $A = 1$ be the sole scalar BNT representative, and take two raw copies with weights 1 and 1/2. The one-letter tensor then has local virtual matrix

$$K^{00} = \mathrm{diag}(1, 1/2). \tag{42}$$

This is a BNT canonical presentation: both weights have modulus at most one, and the first has modulus one. Every positive-length contraction is $1 + 2^{-N} > 0$, the scalar representative is normal and non-nilpotent, and the singleton family has simultaneous one-letter span. Condition (iv) of Theorem 4.9 in Ref. [CPGSV16] also holds: the representative generates MPDOs, the distinct-layer equation is vacuous, and the scalar physical-sector factorization has a positive neighboring operator and normalized rank-one trace coefficients.

The sole local matrix of $K^{[2]}$ is $(K^{00})^2$. If the blocked tensor satisfied Definition 4.1 of Ref. [CPGSV16], its physical-trace transfer would be idempotent, so

$$(K^{00})^4 = (K^{00})^2. \tag{43}$$

Taking traces would give

$$1 + 2^{-4} = 1 + 2^{-2}, \tag{44}$$

which is false. The declaration in `TNLean/MPS/MPDO/Theorem49RepeatedCopyCounterexample.lean`, `MPOTensor.CaseIIAbsorptionCounterexample`

.printed_theorem49_iv_to_v_is_false, kernel-checks explicit component witnesses for all standing data, MPDO positivity of the blocked tensor, and the failure in Eq. (44) in one conjunction. Thus the literal implication (iv)⇒(v) in Theorem 4.9 of Ref. [CPGSV16] is refuted and classified as an *unfaithful statement*. The printed route exhibits *proof-path drift*: condition (iv) does not constrain repeated-copy weights, so no outer-sector assembly can repair the claim.

The viable source route is instead (ii)⇒(v). In the earlier Case-II argument, lines 1646–1665 of Ref. [CPGSV16] use literal zero correlation length to make all copy weights within each BNT sector equal and absorb their common value. The repeated-copy tensor in Eq. (42) fails condition (ii), because $\mathrm{diag}(1, 1/2)$ is not idempotent. Consequently, it does not contradict Proposition prop2to5 of Ref. [CPGSV16]. The remaining work is either the printed per-representative factorization or another construction of the same blocked channel pairs, followed by their projector-controlled combination after common-weight absorption.

7 The vertical one-label identity

The same scale bookkeeping appears in the algebraic form of Theorem 4.14 of Ref. [CPGSV16]. Its length-one condition is

$$m_\gamma = \sum_{\alpha,\beta} c^{(1)}_{\alpha,\beta,\gamma} m_\alpha m_\beta, \tag{45}$$

where

$$m_\alpha = \mathrm{tr}(\mu_\alpha). \tag{46}$$

These formulas appear at lines 972–993 of Ref. [CPGSV16]. In the displayed one-label dimer presentation,

$$c^{(1)}_{0,0,0} = \frac{32}{25}, \tag{47}$$

$$m_0 = \frac{25}{32}. \tag{48}$$

Thus the nonzero solution is fixed by $(c^{(1)}_{0,0,0})^{-1}$ rather than by a unit-weight convention. This statement concerns the explicit presentation and is not an invariance theorem for coefficients under changes of presentation.

8 Horizontal periodic equality does not fix the vertical normalization

The distinction between a horizontal periodic family and a chosen vertical presentation is already visible for one physical letter. Define

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{49}$$

$$X = \begin{pmatrix} 1 & -3/4 \\ 0 & 1 \end{pmatrix}, \tag{50}$$

$$Q = XPX^{-1} = \begin{pmatrix} 1 & 3/4 \\ 0 & 0 \end{pmatrix}. \tag{51}$$

Let M_P and M_Q be the one-letter MPO tensors with sole matrices P and Q, respectively. Since P and Q are idempotent and have trace one, every positive length N satisfies

$$\rho_N(M_P) = \rho_N(M_Q) = 1. \tag{52}$$

Thus both tensors generate MPDOs and have the same unnormalized periodic operator family. They are related by the nonunitary horizontal gauge X, which belongs to the general MPV gauge freedom described at lines 187–199 of Ref. [CPGSV16].

Their normalized one-label vertical representatives differ:

$$A_P = P, \tag{53}$$

$$m_P = 1, \tag{54}$$

$$A_Q = \frac{4}{5}Q, \tag{55}$$

$$m_Q = \frac{5}{4}. \tag{56}$$

Both are normal bond-one tensors, and each gives a positive one-label algebra clause. Their products satisfy

$$A_P^2 = A_P, \tag{57}$$

$$A_Q^2 = \frac{4}{5}A_Q. \tag{58}$$

Consequently, their structure coefficients are

$$c_P^{(L)} = 1, \tag{59}$$

$$c_Q^{(L)} = \left(\frac{4}{5}\right)^L. \tag{60}$$

A singleton label set has only one relabelling, so these coefficients cannot agree after relabelling. The first family is length independent, whereas the second is not.

The declarations in `TNLean/MPS/MPDO/VerticalCoefficientPresentationCounterexample.lean` prove the gauge relation, MPDO positivity, both tensor-attached algebra clauses, Eqs. (59) and (60), and the failure of relabelled equality. In particular, `MPOTensor.VerticalCoefficientPresentationCounterexample.sameMPVPos_and_no_relabelled_coefficient_equality` refutes the theorem proposed in GitHub issue #6395 under its stated bare positive-length MPV-equality hypothesis.

This example is not a counterexample to Proposition 4.13 or Theorem 4.14 of Ref. [CPGSV16]. The source starts with one horizontally canonical MPDO tensor and then chooses its vertical decomposition [CPGSV16, lines 943–991]. No horizontal canonical-form assertion is made here for the sheared tensor M_Q. The source comparison in Appendix C.4 retains the positive ratio $m_\alpha/n_{\sigma(\alpha)}$ instead of discarding the normalization scale [CPGSV16, lines 1997–2008]. Consequently, the source's fixed-presentation characterization and the project's explicit length-dependent RFP example remain unchanged. This counterexample does not establish rigidity between two literal horizontal canonical presentations.

A scale-covariant statement requires explicit vertical transport. Suppose there are a label equivalence e, nonzero scalars s_α, and algebra isomorphisms Φ_L such that every label α satisfies

$$O'_L(\alpha) = s_\alpha^L \Phi_L(O_L(e\alpha)). \tag{61}$$

At a positive length where the target operators are linearly independent, uniqueness of the product expansion gives

$$c'^{(L)}_{\alpha,\beta,\gamma} = \left(\frac{s_\alpha s_\beta}{s_\gamma}\right)^L c^{(L)}_{e\alpha,e\beta,e\gamma}. \tag{62}$$

Exact equality therefore requires a normalization condition such as $s_\alpha s_\beta = s_\gamma$ on every active product triple. Deriving Eq. (61) from two horizontal canonical presentations, and extending Eq. (62) to every positive length, are separate statements not supplied by bare horizontal periodic equality.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.