Nilpotent Basis Elements in the Simple-Tensor Definition of Cirac–Pérez-García–Schuch–Verstraete

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Verdict: local-correction (resolved).

1 Source definition

At line 822 of Ref. [CPGSV16], Cirac–Pérez-García–Schuch–Verstraete define simplicity as follows: "A tensor generating MPDO's is simple if none of the elements in a BNT is nilpotent." Line 819 of the same source defines an element $\mathcal{B}_k$ of the basis of normal tensors to be nilpotent when $\mathrm{tr}_R(\rho^{(N)}(M_k)) = 0$ after tracing out $R \leq D$ sites, for every sufficiently large N.

2 Formal reading

Let $A = \mathcal{B}_k$ be a normal BNT representative with virtual dimension D, and define its physical-trace transfer matrix by

$$\mathcal{T}_A = \sum_i A^{ii}. \tag{1}$$

Here $\mathrm{tr}_R(\rho^{(N)}(A)) = 0$ means that the reduced density operator is zero after tracing out R consecutive sites. For physical multi-indices $\mathbf{i} = (i_1, \ldots, i_{N-R})$ and $\mathbf{j} = (j_1, \ldots, j_{N-R})$, its matrix entries are

$$\left(\mathrm{tr}_R\Big(\rho^{(N)}(A)\Big)\right)_{\mathbf{i},\mathbf{j}} = \mathrm{tr}\big(A^{i_1 j_1} \cdots A^{i_{N-R} j_{N-R}} \mathcal{T}_A^R\big)\,. \tag{2}$$

Choose a sufficiently large N for which the length-$(N-R)$ products of the normal representative span the full virtual matrix algebra. The trace pairing on that algebra is nondegenerate. Consequently,

$$\mathrm{tr}_R\Big(\rho^{(N)}(A)\Big) = 0 \text{ for every sufficiently large } N \iff \mathcal{T}_A^R = 0. \tag{3}$$

Since the nilpotency index of a nilpotent $D\times D$ matrix is at most D, the line-819 condition of Ref. [CPGSV16] is precisely nilpotency of $\mathcal{T}_A$.

The formal predicate `MPOTensor.IsSimple` existentially quantifies over the positive physical blocking required at source line 815 [CPGSV16] and chooses a BNT sector presentation of the blocked doubled-index tensor. Every representative has a positive number of copies, every copy has nonzero weight, the representatives are normal and eventually linearly independent, and the blocked tensor has the same positive-length matrix product vectors as their weighted direct sum. The predicate records non-nilpotency for every representative. The nonzero copy weights implement the nonzero-coefficient convention recorded in `docs/paper-gaps/cpsv16_bnt_uniqueness_zero_coefficient.tex`, without imposing the unit-modulus normalization from source line 246 [CPGSV16].

Positive-length nontriviality is a theorem rather than a defining assumption. A nonzero copy-weight power sum cannot vanish eventually. At a sufficiently large length where one sector coefficient is nonzero, eventual linear independence makes the blocked matrix product vector nonzero. Physical unblocking then gives `MPOTensor.IsSimple.exists_mpo_ne_zero`. Isolated vanishing lengths remain possible. Definition 4.7 of Ref. [CPGSV16] does not require the generated MPO to be nonzero at every positive chain length, and the formal predicate imposes no such requirement.

The predicate `MPOTensor.IsSimpleCanonicalForm` is a separate, normalized, fixed-representative condition on an already-blocked tensor. It is used for the Appendix C.2 argument of Ref. [CPGSV16] and includes a normalized sector decomposition with the source's line-246 unit-weight convention. No equivalence with `MPOTensor.IsSimple` is asserted.

The toric-code boundary tensor discussed after Theorem 4.9 of Ref. [CPGSV16] confirms this reading. Its second basis element has the one-dimensional transfer matrix

$$\mathcal{T} = 0. \tag{4}$$

The tensor is therefore not simple, as required by the source's treatment of that example outside the simple case.

The source states the predicate for "a BNT" of the blocked tensor and then asserts that BNTs are unique up to gauge, phase, and permutation [CPGSV16]. The theorem `MPSTensor.IsBNTSectorPresentation.equiv_of_sameMPVPos` matches two presentations, and `MPOTensor.bnt_basis_not_isNilpotent_iff` transports nilpotency through the resulting permutation, virtual-dimension identification, gauge transformation, and phase. Thus the choice of presentation is immaterial. The predicate does not require a normalized horizontal canonical-form presentation.

3 Classification

The formal predicate is complete for source lines 217–246 and Definition 4.7 of Ref. [CPGSV16], under the identification in Eq. 3 of the line-819 partial-trace criterion with matrix nilpotency of the physical-trace transfer. The predicate requires a nonempty BNT sector presentation, and positive-length nontriviality follows from that presentation. It does not require nonvanishing at every positive length, and the source imposes no such condition.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.