Normalization of the tensor printed in CPSV16 Example 4.12

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Verdict: false-source (open).

1 Printed representative

Example 4.12 of Ref. [CPGSV16] (`Papers/1606.00608/MPDO-22-12-17-2.tex:932--939`) prints a tensor M with physical dimension $d = 2$, virtual dimension $D = 2$, and components

$$1 = M^{00}_{00} = M^{11}_{00} = M^{00}_{11} = -M^{11}_{11}, \tag{1}$$

with all other components equal to zero. In matrix notation,

$$M^{00} = \mathbb{1}, \tag{2}$$
$$M^{11} = \sigma_z, \tag{3}$$
$$M^{01} = M^{10} = 0, \tag{4}$$

where

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{5}$$

Let $\rho^{(N)}(M)$ denote the periodic length-N operator obtained by closing the virtual indices of M. For every $N > 0$, the printed tensor gives

$$\rho^{(N)}(M) = \mathbb{1}^{\otimes N} + \sigma_z^{\otimes N}. \tag{6}$$

Its trace is

$$\mathrm{tr}\Big(\rho^{(N)}(M)\Big) = 2^N, \tag{7}$$

not 1. Thus the printed M is an unnormalized tensor representative, although division of the resulting positive operator by 2^N gives a normalized density operator.

2 The fixed-representative mismatch

Define the physical-trace transfer matrix of the printed tensor by $\mathcal{T}_M = M^{00} + M^{11}$. Equations 2 and 3 give

$$\mathcal{T}_M = M^{00} + M^{11} = \mathbb{1} + \sigma_z = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

Consequently,

$$\mathcal{T}_M^2 = 2\mathcal{T}_M, \tag{9}$$

$$\mathcal{T}_M^2 \neq \mathcal{T}_M. \tag{10}$$

Equation 9 proves the project's scale-invariant relation with scale 2, but Eq. 10 contradicts the literal zero-correlation-length equation in Definition 4.2 of Ref. [CPGSV16]. The trace-preserving maps $\mathcal{S}$ and $\mathcal{T}$ in Definition 4.1 of Ref. [CPGSV16] also force equality of the one-site and two-site traces for every virtual boundary. Equivalently, they require literal idempotence of $\mathcal{T}_M$. The printed representative therefore cannot satisfy that fixed-tensor renormalization condition.

This failure does not contradict the normalized density family. Define the rescaled representative by

$$\widehat{M} = \frac{1}{2}M. \tag{11}$$

Its transfer matrix satisfies

$$\mathcal{T}_{\widehat{M}} = \frac{1}{2}\mathcal{T}_M, \tag{12}$$

$$\mathcal{T}_{\widehat{M}}^2 = \mathcal{T}_{\widehat{M}}. \tag{13}$$

The periodic family of $\widehat{M}$ is

$$\rho^{(N)}(\widehat{M}) = 2^{-N}\left(\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N}\right). \tag{14}$$

This is precisely the normalized density family associated with the printed tensor. The representative $\widehat{M} = (1/2)M$ is therefore the likely tensor intended by the assertion at source line 938 that trace-preserving maps exist [CPGSV16]. The paper does not include this factor in the printed tensor components or explicitly change representatives before making that assertion.

3 Formalization boundary

The current formalization keeps the two representatives separate. For the literal printed tensor M, it proves the exact positive-length formula, positivity, one-site loss of the σ_z term, saturation of the area law, zero correlation length only in the project's scale-invariant sense, explicit failure of literal zero correlation length, and failure of the fixed-representative trace-preserving renormalization condition.

For $\widehat{M} = (1/2)M$, define the parity function $\pi(a,b) = a \oplus b$ and the Kraus operators

$$C_{a,b} = |\pi(a,b)\rangle\langle a,b|, \tag{15}$$

$$R_{a,b} = 2^{-1/2}|a,b\rangle\langle \pi(a,b)|. \tag{16}$$

Each parity fibre has two elements. Therefore,

$$\sum_{a,b} C_{a,b}^{\dagger} C_{a,b} = \mathbb{1}_4, \tag{17}$$

$$\sum_{a,b} R_{a,b}^{\dagger} R_{a,b} = \mathbb{1}_2. \tag{18}$$

The corresponding completely positive maps are trace preserving. The local tensor products satisfy

$$\widehat{M}^{aa}\widehat{M}^{bb} = \frac{1}{2}\widehat{M}^{\pi(a,b)\pi(a,b)}, \tag{19}$$

and the off-diagonal physical letters vanish. Direct substitution therefore proves both channel equations of Definition 4.1 of Ref. [CPGSV16] for every virtual boundary matrix. These results are recorded by `MPOTensor.CPSVExample412NormalizedRFP.Mhat`, `MPOTensor.CPSVExample412NormalizedRFP.mpo_Mhat_eq_normalizedMPO_M`, `MPOTensor.CPSVExample412NormalizedRFP.Mhat_isMPDO`, and `MPOTensor.CPSVExample412NormalizedRFP.Mhat_isRFPViaTS`.

This conclusion concerns only the two channel equations. It does not resolve a second normalization tension in the source. As a doubled-index matrix product state, M has two bond-one sectors with physical vectors $(1,0,0,1)$ and $(1,0,0,-1)$. Division by $\sqrt{2}$ gives their normal representatives. Consequently, the two horizontal normal-block weights of $\widehat{M}$ are both $1/\sqrt{2}$. Neither has modulus one, whereas source line 246 requires at least one unit-modulus weight [CPGSV16]. The proved bare channel predicate must therefore not be presented as a proof that $\widehat{M}$ satisfies every standing canonical-form convention in Definition 4.1 of Ref. [CPGSV16]. This boundary is also recorded in `docs/paper-gaps/cpsv16_unit_weight_rfp_scale_tension.tex`.

The independent four-site entropy calculation is also complete. For the normalized state, regroup the sites as $A = \{0\}$, $B = \{1,3\}$, and $C = \{2\}$. The AB and BC marginals are $\mathbb{1}_8/8$, the B marginal is $\mathbb{1}_4/4$, and the full state is uniform on eight even-parity configurations. Thus,

$$S(ABC) = 3\log 2, \tag{20}$$

$$S(B) = 2\log 2, \tag{21}$$

$$S(AB) = 3\log 2, \tag{22}$$

$$S(BC) = 3\log 2. \tag{23}$$

The strong-subadditivity defect is therefore

$$S(AB) + S(BC) - S(B) - S(ABC) = \log 2 > 0. \tag{24}$$

This proves the state-specific Markov obstruction without using the normalized fixed-point maps.

The full GSNNCH exclusion already concerns the normalized family because `MPOTensor.IsGSNNCH` quantifies over `MPOTensor.normalizedMPO`. It is now proved directly for the printed tensor. At four sites, the sector sum in Definition 4.8 of Ref. [CPGSV16] splits into positive factors built from the complementary adjacent-bond products $B_3^{(x)} B_0^{(x)}$ and $B_1^{(x)} B_2^{(x)}$, acting on the three-site windows $(3, 0, 1)$ and $(1, 2, 3)$, respectively. Orthogonality of distinct sectors removes the cross terms in their product. Commutativity within each sector identifies the product of the two factors with the corresponding four-bond sector product. The resulting positive overlapping factors commute and give the quantum Markov decomposition `MPOTensor.nonempty_quantumMarkovDecomposition_fourCycle_of_isGSNNCHAt`, and hence the strong-subadditivity equality `MPOTensor.isSSAEquality_fourCycle_of_isGSNNCHAt`. This derivation uses only Definition 4.8 of Ref. [CPGSV16]. It does not assume zero correlation length, normality, injectivity, primitivity, a single sector, or completeness of the sector projections.

The identity `MPOTensor.CPSVExample412Literal.fourCycleTripartiteState_eq_generic` shows that this general regrouping is exactly the state used in the literal entropy calculation. Consequently,

$$\neg\, \mathrm{IsGSNNCH}(M) \quad \text{holds.} \tag{25}$$

This statement is proved by `MPOTensor.CPSVExample412Literal.M_not_isGSNNCH`. It discharges the missing and formerly unlinked four-cycle task tracked in GitHub issue #6072 and the missing Example 4.12 conclusion tracked in GitHub issue #6045. The earlier attempt to pass through Proposition *prop4to2* had proof-path drift: that proposition concerns a different condition-level implication and is not used here. Transport to the separately named $\widehat{M}$ is unnecessary because the GSNNCH predicate already uses the normalized family.

4 Verdict

The tensor M printed in Example 4.12 of Ref. [CPGSV16] is the formal tensor `MPOTensor.CPSVExample412Literal.M`. Its positive-length operator formula in Eq. 6 is proved by `MPOTensor.CPSVExample412Literal.rho_eq_finKronecker`, and its saturation statement is proved by `MPOTensor.CPSVExample412Literal.M_isSAL`. The transfer calculation in Eq. 9 is formalized by `MPOTensor.CPSVExample412Literal.physTraceTransfer_M_sq`, while `MPOTensor.CPSVExample412Literal.physTraceTransfer_M_not_idempotent` proves Eq. 10. Therefore, `MPOTensor.CPSVExample412Literal.M_not_isRFPViaTS` proves that the fixed-representative renormalization assertion is false for the tensor exactly as printed.

The normalized tensor $\widehat{M} = (1/2)M$ is the likely local correction. Its periodic family is identified with the normalized periodic family of M by tensor-level scalar rescaling and `MPOTensor.mpo_smul`; it is not an independent tensor presentation. The parity Kraus families in Eqs. 15 and 16 prove the bare trace-preserving channel predicate without additional hypotheses, discharging the mathematical task tracked in GitHub issue #6044. They do not repair the incompatible source line-246 unit-weight convention. The conclusion therefore remains separated into the false literal claim, the true normalized channel equations, the unresolved unit-weight boundary, and the proved full GSNNCH exclusion.

For the normalized four-site state, `MPOTensor.CPSVExample412Literal.fourCycleTripartiteState_not_isSSAEquality` proves the strict defect in Eq. 24. Its proof uses the exact marginals `MPOTensor.CPSVExample412Literal.traceC_fourCycleTripartiteState`, `MPOTensor.CPSVExample412Literal.traceA_fourCycleTripartiteState`, and `MPOTensor.CPSVExample412Literal.traceAC_fourCycleTripartiteState`, together with their entropy calculations. This discharges the mathematical task tracked in GitHub issue #6073. The general four-cycle Markov implication and the exact coordinate identity then give `MPOTensor.CPSVExample412Literal.M_not_isGSNNCH`, discharging the tasks tracked in GitHub issue #6072 and GitHub issue #6045 without hypotheses beyond the source GSNNCH definition.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.