The marginal-support expansion in CPSV16 Appendix D.2

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Verdict: local-correction (resolved).

Let $\mathcal{H}_A$, $\mathcal{H}_X$, and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces. Let $K_{AXB} \subseteq \mathcal{H}_A \otimes \mathcal{H}_X \otimes \mathcal{H}_B$, and let P_{AXB} be the orthogonal projector onto K_{AXB}. Define the reduced operators by

$$\rho_{AX} = \mathrm{tr}_B(P_{AXB}), \tag{1}$$

$$\rho_{XB} = \mathrm{tr}_A(P_{AXB}). \tag{2}$$

The two marginal supports are $\mathrm{ran}(\rho_{AX})$ and $\mathrm{ran}(\rho_{XB})$, with orthogonal projectors P_{AX} and P_{XB}, respectively. For an operator on AX or XB, a hat denotes its lift to the full tripartite space. In particular,

$$\widehat{P}_{AX} = P_{AX} \otimes \mathbb{1}_B, \tag{3}$$

$$\widehat{P}_{XB} = \mathbb{1}_A \otimes P_{XB}. \tag{4}$$

The same convention defines $\widehat{\rho}_{AX}$ and $\widehat{\rho}_{XB}$.

1 The source step

In the proof of Proposition D.3 of Ref. [CPGSV16, lines 2236–2242], the source chooses orthonormal bases $\{\alpha_i\}_i$ and $\{\beta_i\}_i$ of the two marginal supports and asserts that each basis vector is a single partial contraction:

$$\alpha_i = \langle \widetilde{b}_i | \varphi_{b,i}, \tag{5}$$

$$\beta_i = \langle \widetilde{a}_i | \varphi_{a,i}, \tag{6}$$

where $\varphi_{a,i}, \varphi_{b,i} \in K_{AXB}$ and the contractions act on the corresponding outer tensor factors.

The support of a reduced operator is spanned by partial contractions, but an arbitrary vector in that span need not equal one partial contraction. Thus Eqs. 5 and 6 do not follow from the definition of marginal support.

2 Local correction

The subsequent matrix-unit argument needs only a factorization of each marginal support projector through the corresponding reduced operator. Fix orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, and define the elementary matrix units on the outer factors by

$$E^A_{ak} = |a\rangle\langle k|, \tag{7}$$
$$E^B_{bk} = |b\rangle\langle k|. \tag{8}$$

When these matrix units occur in products with P_{AXB}, they are lifted by the identity on the other tensor factors. The reduced operators then have the exact expansions

$$\widehat{\rho}_{AX} = \sum_{b,k} E^B_{bk} P_{AXB} E^B_{kb}, \tag{9}$$

$$\widehat{\rho}_{XB} = \sum_{a,k} E^A_{ak} P_{AXB} E^A_{ka}. \tag{10}$$

Define $g(0) = 0$ and $g(x) = x^{-1}$ for $x \neq 0$. Functional calculus gives

$$P_{AX} = g(\rho_{AX})\rho_{AX} = \rho_{AX} g(\rho_{AX}), \tag{11}$$
$$P_{XB} = g(\rho_{XB})\rho_{XB} = \rho_{XB} g(\rho_{XB}). \tag{12}$$

The decorrelation identity first gives

$$\widehat{\rho}_{XB}(\mathbb{1} - P_{AXB})\widehat{\rho}_{AX} = 0, \tag{13}$$
$$\widehat{\rho}_{AX}(\mathbb{1} - P_{AXB})\widehat{\rho}_{XB} = 0. \tag{14}$$

Multiplication by the reciprocal-on-support factors from Eqs. 11 and 12 yields

$$\widehat{P}_{XB}(\mathbb{1} - P_{AXB})\widehat{P}_{AX} = 0, \tag{15}$$
$$\widehat{P}_{AX}(\mathbb{1} - P_{AXB})\widehat{P}_{XB} = 0. \tag{16}$$

The absorption identities then imply

$$\widehat{P}_{XB}\widehat{P}_{AX} = P_{AXB}, \tag{17}$$
$$\widehat{P}_{AX}\widehat{P}_{XB} = P_{AXB}. \tag{18}$$

This correction retains the matrix-unit and marginal-support argument of Ref. [CPGSV16], but replaces the unsupported single-contraction assertion by the valid factorizations in Eqs. 11 and 12. It adds no hypothesis to Proposition D.3 of Ref. [CPGSV16].

3 Formal statement

The reduced-operator expansions in Eqs. 9 and 10 are formalized by `TripartiteDecorrelation.lift_marginalAX_eq_sum` and `TripartiteDecorrelation.lift_marginalXB_eq_sum`. The complementary support products in Eqs. 15 and 16 are formalized by `TripartiteDecorrelation.IsDecorrelated.supports_mul_complement_mul_eq_zero` and its reverse-order counterpart. The product identities in Eqs. 17 and 18 are assembled by `TripartiteDecorrelation.supportProducts_eq`.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.