The Nonzero-Coefficient Convention for CPSV16 Canonical Forms

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Verdict: local-correction (resolved).

1 The convention

Cirac–Pérez-García–Schuch–Verstraete construct the canonical form by listing the invariant blocks of the completely positive map and normalizing each block separately [CPGSV16, lines 214–225]. The construction therefore produces precisely the blocks that occur in the tensor. Equation `eq:II_CF1` of Ref. [CPGSV16, lines 238–244] defines

$$A^i = \bigoplus_{k=1}^{r} \mu_k A_k^i. \tag{1}$$

The displayed definition does not repeat that every μ_k is nonzero. However, the subsequent normalization $|\mu_k| \leq 1$, with at least one coefficient equal to one, presupposes this condition [CPGSV16, line 246]: a summand with coefficient zero contributes nothing to any positive-length matrix product vector and is not a block of the decomposition.

The formalization reads the canonical form with every coefficient μ_k nonzero. This requirement is part of the canonical-form data.[1] A basis of normal tensors in the sense of Definition 2.4 of Ref. [CPGSV16] is a family of normal tensors whose matrix product vectors span those of A at every positive length and are eventually linearly independent. After gauge-phase-equivalent blocks are identified, the blocks of a canonical form constitute such a basis, and every basis element occurs with nonzero coefficient.

2 Consequences

Under this convention, the following source statements are formalized as stated, with the indicated blueprint nodes.

[1] Formal field: `MPSTensor.CPSVCanonicalFormData.weights_ne_zero` in `TNLean/MPS/CanonicalForm/Definitions.lean`.

- Proposition 2.7 of Ref. [CPGSV16, lines 278–280 and 1135–1146] states that a family is a basis of normal tensors for a canonical form if and only if every representative is normal, every block is gauge-phase equivalent to a representative, and distinct representatives are not gauge-phase equivalent. The corresponding blueprint node is `thm:cpsv_bnt_characterization`.[2]

- The uniqueness statement in Ref. [CPGSV16, line 1148] says that two bases of normal tensors for the same positive-length matrix product vector family admit a bijective gauge-phase matching. It follows by applying Proposition 2.7 of Ref. [CPGSV16] in both directions. The statement is formalized for basis-of-normal-tensors sector presentations, in which every basis element carries a nonzero coefficient. The corresponding blueprint node is `thm:cpsv_bnt_presentation_uniqueness`.[3]

  The literal Definition 2.4 family predicate[4] remains available as an interface and carries no uniqueness claim. For a one-block tensor A_0 with coefficient 1, both (A_0) and (A_0, A_1) satisfy that family predicate if the coefficient of A_1 is set to zero at every length. The second family contains a zero-coefficient member. The convention excludes that member from every basis presentation, so this construction is not a counterexample to the formalized uniqueness statement.

- The proportional and equal fundamental theorems and their unitary refinements are Theorem 2.10, Corollary 2.11, and Corollary A.6 of Ref. [CPGSV16, lines 349–361 and 1167–1199]. Their blueprint nodes are `thm:cpgsv_multiblock_ft_source`, `thm:cpgsv_equal_case_source`, `cor:sector_bnt_proportional_unitary_sector_match`, and `cor:sector_bnt_equal_unitary_global_gauge`.

- Lemma A.5 of Ref. [CPGSV16, lines 1155–1163] states that finite-range power sums of nonzero lists determine equal lengths and equal multisets. The lemma applies to the nonzero coefficients of a canonical form. Its blueprint node is `thm:bounded_power_sum_multiset`.

- Corollary 3.12 of Ref. [CPGSV16, lines 583–590] states that every block of a canonical-form renormalization fixed point has a unit-modulus coefficient and an idempotent transfer map, and that the phase-class representatives have a joint residual isometry. The corresponding blueprint nodes are `thm:cpsv_weight_norm_one_and_block_rfp` and `thm:cpsv_phase_class_residual_isometries`.

Without the nonzero-coefficient convention, one could adjoin a normal tensor with coefficient identically zero to any basis without changing any positive-length matrix product vector. The cardinality conclusions of the uniqueness

[2] Formal declaration: `MPSTensor.isCPSVBasisOfNormalTensors_iff_canonicalForm_covered_and_minimal`.
[3] Formal declaration: `MPSTensor.IsBNTSectorPresentation.equiv_of_sameMPVPos`.
[4] Formal declaration: `MPSTensor.IsCPSVBasisOfNormalTensors`.

statement, Theorem 2.10, and Lemma A.5 of Ref. [CPGSV16] would then fail. The convention records the source's intended reading and makes the formal definition consistent with Equation `eq:II_CF1` of Ref. [CPGSV16], reproduced in Eq. 1.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.