The Nonnegativity Hypothesis in the Small-Overlaps Bound

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1 The assertion

This note concerns the bound on the sum over small overlaps in the rank-reduction step of the orthogonalization lemma, [JNV+20, Section *Orthogonalization lemma*]. [1] We use the notation of the cited source: the overlaps $o_{a,i}$ are defined by

$$o_{a,i} = \langle \psi, (P_{v_{a,i}} \otimes I)\psi \rangle,$$

where ψ is the state and $P_{v_{a,i}}$ is the rank-one projection onto the unit vector $v_{a,i}$ from the range of a projector. The total number of overlap pairs is r, the set Small is the complement of the d largest overlaps, and ζ is the self-consistency error parameter with $0 \leq \zeta \leq 1/4$.

At this point the paper asserts the estimate

$$\sum_{(a,i)\in\mathsf{Small}} o_{a,i} \leq 4\sqrt{\zeta}, \qquad (\texttt{eq:small-overlaps})$$

using the auxiliary bounds

$$|\mathsf{Small}| \leq 2\sqrt{\zeta} \cdot r, \qquad \sum_{a,i} o_{a,i} \leq 1 + 2\sqrt{\zeta},$$

together with the fact that every overlap in Small is at most every overlap outside Small. The proof of the auxiliary bounds uses the overlaps themselves; they are manifestly non-negative because each is an expectation of a positive operator against a state.

[1] The discrepancy recorded here was identified in GitHub issue #938. In the TeX source consulted here, the estimate occurs in `references/ldt-paper/orthonormalization.tex`, lines 606–612.

2 The abstraction in Lean

The formal development, following the blueprint, extracts the combinatorial content of the estimate and proves it for an arbitrary real-valued function, without assuming nonnegativity. [2]

In the statement of the lemma, the overlaps are replaced by a function $f\colon \alpha \to \mathbb{R}$ on a finite set α. The subset $L \subseteq \alpha$ plays the role of the large overlaps, and L^{c} plays the role of Small. The hypotheses are $|L| = d$, $d < |\alpha|$, $|\alpha| \le (1+2\sqrt{\zeta})\, d$, the ordering condition $f(s) \le f(l)$ for all $s \in L^{\mathsf{c}}$ and $l \in L$, the total estimate $\sum_{x\in\alpha} f(x) \le 1 + 2\sqrt{\zeta}$, and $0 \le \zeta \le 1/4$. The conclusion is

$$\sum_{s\in L^{\mathsf{c}}} f(s) \le 4\sqrt{\zeta}.$$

The proof does not need the hypothesis that every $f(x)$ is non-negative. It uses only the ordering hypothesis, the total estimate, and the cardinality relations; the sign of f plays no role in the calculation.

3 Comparison

In the paper, the values $o_{a,i}$ are non-negative by construction. Thus any abstract version which retains all paper-specific facts may include the hypothesis $f(x) \ge 0$. The formal lemma proves the same inequality without that hypothesis. It is therefore stronger than the natural abstracted statement with non-negative f, while leaving the concrete estimate in the paper unchanged.

The difference does not affect soundness. The formal proof of the orthogonalization lemma instantiates the combinatorial lemma with the actual overlaps $o_{a,i}$, which are indeed non-negative. In that instantiation the ordering and total-estimate hypotheses are verified from the properties of the overlaps, and the lemma then supplies the needed bound on the sum over the small entries.

[2] The formal lemma is `sum_small_le_four_sqrt` in `MIPStarRE/LDT/MakingMeasurementsProjective/QXPLayer/TruncationCombinatorics.lean`, lines 164–175. It is accompanied by two auxiliary lemmas `card_mul_sum_small_le` (lines 100–119) and `card_univ_mul_sum_compl_le` (lines 125–148).

4 Relation with the blueprint

The blueprint states the same abstract averaging lemma as `lem:small-overlap-averaging` in [MIP26, Chapter *Making measurements projective*]. [3] It takes $f : \alpha \to \mathbb{R}$ to be an arbitrary real-valued function and does not require $f \geq 0$. Its hypotheses are the ordering condition, the total estimate, the size constraints, and $0 \leq \zeta \leq 1/4$, followed by the same conclusion $\sum_{s \in L^c} f(s) \leq 4\sqrt{\zeta}$. Thus the blueprint and the formal statement agree: both record the ordered double-counting argument in a form which is more general than the concrete overlap calculation in the paper.

5 Conclusion

There is no theorem-level gap. The formal lemma is a conservative strengthening of the argument in the paper. The paper's proof of the small-overlaps bound uses only the ordering hypothesis, the total estimate, and the cardinality relations; it never invokes the nonnegativity of the overlaps except through their definition. The formal development makes this explicit by removing the nonnegativity condition from the statement. In the application within the orthogonalization lemma, the stronger statement is specialized back to the non-negative overlaps, so the formal soundness argument remains faithful to the paper.

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.

[MIP26] MIPStarRE Project. Blueprint for arxiv:2009.12982: Quantum soundness of the classical low individual degree test. `https://sirui-lu.com/MIPStarRE`, 2026. Lean blueprint for the MIPStarRE formalization.

[3] In the blueprint source consulted here, this is in `blueprint/src/chapter/ch04_projective.tex`, lines 419–440.