Resolved Missing ν-Consistency Hypothesis in the Self-Improvement Theorem

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1 At a glance

- **Status: resolved.** The statement-level discrepancy is resolved: the Lean theorem selfImprovement contains the paper's ν-consistency hypothesis. The proof-level gap described in the original note has also been closed: selfImprovement and the induction-section reformulation are now proved from the corrected source-facing hypotheses. Thus issue #1515 is historical for the current Lean tree.
- **Difficulty: low as a statement repair, substantial as the former proof construction.** The missing hypothesis was easy to identify from the paper statement. The later proof closure required the Section 9 helper, semidefinite-programming, orthonormalization, and final-field constructions.
- **Estimated weight:** no remaining proof construction is expected for this note. It remains as documentation of the former statement-level discrepancy and its resolution.
- **Mathlib/project split:** roughly **25% Mathlib** and **75% project-local** for the proof route that closed the issue. Mathlib supplies finite-dimensional operator and matrix order infrastructure; the project supplies the LDT self-improvement interfaces and transports.
- **Key Mathlib inputs:** positive semidefinite matrices, finite sums of matrix-valued effects, trace and operator inequalities, and the finite-dimensional SDP infrastructure used by the Section 9 proof.

2 Key theorem forms

The self-improvement step in [JNV+20, Section 9] takes a measurement $G \in \mathrm{PolyMeas}(m, q, d)$, where m is the number of variables, q the field size, and d the degree bound of the low individual degree test. The surrounding strategy is an $(\epsilon, \delta, \gamma)$-good symmetric strategy; its data includes a projective measurement family $A = \{A^{\boldsymbol{u}}\}_{\boldsymbol{u} \in \mathbb{F}_q^m}$ with outcomes in $\mathbb{F}_q$ and a bipartite state $|\psi\rangle$. The function g maps a point $\boldsymbol{u}$ to its evaluation, and $[g(\boldsymbol{u}) = a]$ selects the outcome indexed by a. The paper's self-improvement theorem then assumes the following consistency hypothesis on G: on average over $\boldsymbol{u} \in \mathbb{F}_q^m$,

$$A_a^{\boldsymbol{u}} \otimes I \simeq_\nu I \otimes G_{[g(\boldsymbol{u})=a]}. \tag{1}$$

Here $\simeq_\nu$ is the paper's approximation relation comparing two families of measurements by the expectation of their difference: for a bipartite state $|\psi\rangle$, the expectation difference is at most ν. The theorem produces an output measurement H and a dual certificate operator Z dominating the averaged A-operators. The parameter ζ is set to $\zeta = 3000\, m\, (\epsilon^{1/32} + \delta^{1/32} + (d/q)^{1/32})$. Under (1) the theorem concludes completeness $\langle\psi| H \otimes I |\psi\rangle \geq (1 - \nu) - \zeta$, point consistency of H, strong self-consistency of H at level ζ, and the operator bound $\langle\psi| Z \otimes (I - H) |\psi\rangle \leq \zeta$.[1]

3 The formal statement

The Lean declaration MIPStarRE.LDT.SelfImprovement.selfImprovement now includes the consistency hypothesis (1). The former obligation bundle SelfImprovementObligations and the conditional theorem that used it have been removed. Thus the discrepancy recorded in this note has been resolved at the level of the public theorem statement.

The proof-level gap recorded by the original version of this note has since been discharged. The current theorem proves the source-shaped statement directly, rather than representing the missing work by an explicit sorry or by extra hypotheses on a conditional substitute for the theorem.

The induction-step theorem MIPStarRE.LDT.MainInductionStep.selfImprovementInInductionSection has the same status: its statement includes the input consistency hypothesis, and the Section 9 work needed by that reformulation is now checked.

4 Mathematical consequence

The previous theorem statement was too weak to express the paper's claim, because ν was not related to the input measurement. The present theorem statement no longer has that defect. The helper strong self-consistency estimate, the orthonormalization step, and the final-field transport are now derived inside the checked proof rather than supplied as auxiliary source hypotheses.

[1] The paper passages are `references/ldt-paper/self_improvement.tex`, lines 635–671 (theorem) and lines 24–60 (helper lemma); the corresponding induction-section theorem is in `references/ldt-paper/inductive_step.tex`, lines 249–286. This note is part of the retrospective audit in GitHub issue #930. The discrepancy is tracked for resolution in GitHub issue #931.

5 Blueprint status

The blueprint entry now points to the theorem selfImprovement. No top-level conditional obligation theorem is recorded as the formalization of the paper theorem. The former proof-level gap tracked by issue #1515 is historical for the current code.

6 Error coefficients

The paper's helper lemma carries its own error parameter $\widehat{\zeta} = 100\,m\,(\epsilon^{1/2} + \delta^{1/2} + (d/q)^{1/2})$; the orthonormalization step contributes a further error $\widehat{\zeta}_{\mathrm{ortho}} = 100\,\widehat{\zeta}^{\,1/4}$. All error constants match the paper when the obligations are discharged. The helper error matches $\widehat{\zeta}$; the orthogonalization error delegates to the orthonormalization infrastructure, which uses the paper's $100\widehat{\zeta}^{\,1/4}$; the data-processing error $8\widehat{\zeta}+8\sqrt{\widehat{\zeta}_{\mathrm{ortho}}}$ matches the paper; and the final self-improvement error $\zeta = 3000\,m\,(\epsilon^{1/32} + \delta^{1/32} + (d/q)^{1/32})$ matches the paper.

7 Conclusion

The paper's self-improvement theorem requires an approximation hypothesis $\simeq_\nu$ linking the input measurement G to the parent measurement A. The Lean statement now contains this hypothesis, and the corresponding proof is checked without reintroducing the missing work as auxiliary hypotheses.

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.