The First Successor-Step Error Absorption in the Main Induction

2026-05-01

1 The coefficient estimate in the source proof

We examine the last scalar absorption in the proof of the main induction theorem of [JNV$^+$20, Section *The main induction step*].[1] In the successor step, the symbol m denotes the dimension of the restricted slices, so the theorem being proved is the case of dimension $m+1$. After self-improvement and pasting, the proof obtains an error parameter σ^* and an intermediate parameter ν such that

$$\sigma^* \leq \left(1 + \frac{1}{100m}\right)(m^2+3)\left(\nu + e^{-k/(80000m^2)}\right). \tag{eq:gonna-bound-m-function}$$

The source then says that, because $m \geq 2$,

$$\left(1 + \frac{1}{100m}\right)(m^2+3) \leq (m+1)^2. \tag{successor-coefficient}$$

This closes the comparison with the target bound in dimension $m+1$.

2 The first successor step

The induction begins with the base case $m = 1$ and then applies the successor step from dimension 1 to dimension 2. In that first successor step,

[1] This note was found during the retrospective audit tracked by GitHub issue #930. The source passage is `references/ldt-paper/inductive_step.tex`, lines 584–620. The separate large-k side condition in the same theorem is documented in `docs/paper-gaps/issue-906-main-formal-k-bound.tex`; it is not the point at issue here.

the slice dimension in the displayed coefficient is $m = 1$. The coefficient comparison above is then false:

$$\left(1 + \frac{1}{100}\right)(1^2 + 3) = \frac{101}{25} > 4 = (1 + 1)^2.$$

Thus the printed reason "because $m \geq 2$" does not justify the first successor step. This is independent of the large-k side condition: increasing k does not change this coefficient comparison.

The issue is only in this final scalar absorption. It does not affect the base case, the restricted-probability estimates, the self-improvement input, or the pasting theorem statement. It also does not show that the theorem is false in dimension 2; it shows that the displayed coefficient comparison is not the argument that proves that case.

3 The formal replacement argument

The formal proof keeps the same theorem statement and the same final error parameter, but uses a sharper intermediate estimate before the last coefficient comparison.[2] Let $E_m = e^{-k/(80000m^2)}$. The pasting parameter which the source bounds by ν is in fact bounded, in the small-parameter branch of the induction, by $\nu/5$. Together with the bound $\zeta \leq \nu$ for the averaged self-improvement error, this gives

$$\begin{aligned}
\sigma^* &\leq \left(m^2(\nu + E_m) + \nu\right)\left(1 + \frac{1}{100m}\right) + \frac{2}{5}\nu + E_m \\
&= \left((m^2 + 1)\left(1 + \frac{1}{100m}\right) + \frac{2}{5}\right)\nu + \left(m^2\left(1 + \frac{1}{100m}\right) + 1\right)E_m.
\end{aligned}$$

For every $m \geq 1$,

$$(m^2 + 1)\left(1 + \frac{1}{100m}\right) + \frac{2}{5} \leq (m + 1)^2, \qquad m^2\left(1 + \frac{1}{100m}\right) + 1 \leq (m + 1)^2.$$

The two coefficients are therefore separately absorbed into the target coefficient $(m + 1)^2$, including at $m = 1$. Finally $E_m \leq e^{-k/(80000(m+1)^2)}$, so the desired successor-step bound follows.

[2]The sharper estimate is the private lemma ldPastingInInductionNu_le_fifth_mainInductionNu; the final assembly is assembleAveragedPastingInput. Both are in `MIPStarRE/LDT/MainInductionStep/Theorems.lean`.

The proof also has the usual saturated-error branch: if the target error is at least 1, a trivial measurement gives the conclusion. The sharper absorption above is used only in the nontrivial branch where the small-parameter inequalities are derived.

4 Comparison with the blueprint and formal development

The blueprint now records this sharper absorption argument in the proof sketch for the main induction theorem, rather than repeating the source coefficient comparison with the assumption $m \geq 2$.

The formal development does not strengthen the theorem statement to avoid the first successor step. Instead it proves the sharper one-fifth bound for the pasting parameter and uses the two coefficient inequalities above. This is a replacement for the final scalar absorption in the source proof, not a change to the conclusion of the main induction theorem.

5 Conclusion

The source proof contains a harmless gap in the final scalar absorption of the successor step. The displayed coefficient estimate using $(1 + 1/(100m))\,(m^2 + 3)$ requires $m \geq 2$, while the induction also needs the successor step with $m = 1$. The formal proof closes the missing first successor step by using the stronger bound that the pasting error parameter is at most $\nu/5$, and then absorbing the ν and exponential coefficients separately. The theorem statement and final constants remain unchanged.

References

[JNV$^+$20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.