Issue 904: The Missing Completion Term in the Definition of ζ_2

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1 At a glance

- **Difficulty.** Low. The issue is a scalar bookkeeping discrepancy in the completion step, not a missing operator-theoretic construction.
- **Estimated weight.** One local scalar calculation and the downstream replacement of the affected cascade constant.
- **Mathlib/project split.** Mostly project-local; Mathlib supplies only the elementary real inequalities used after the project-specific completion estimate has been instantiated.
- **Key mathematical inputs.** The completion-to-measurement proposition, the orthogonalization error estimate, and the square-root absorption into the definition of ζ_2.

2 Key theorem forms for the completion step

This note concerns the scalar in the completion estimate in [JNV$^+$20, Section *The main induction step*]. [1] We write $X \approx_\eta Y$ for the state-dependent squared-distance relation, with respect to the shared state ψ fixed in the surrounding induction argument, and with error parameter η. The symbol I denotes the identity operator on the tensor factor which is not being measured.

[1] The issue record is GitHub issue #904; the corresponding repair is GitHub pull request #909. In repository notation, these are GitHub issue #904 and PR #909. In the TeX source consulted here, the estimate occurs in `references/ldt-paper/inductive_step.tex`, lines 135–149.

At this point in the induction, G^{A} and G^{B} are the polynomial-outcome measurements before projectivization. Their self-consistency error, obtained in the preceding part of the induction, is denoted by ζ_1. Applying the orthogonalization lemma for measurements to this self-consistency estimate gives, for each polynomial outcome g,

$$G_g^{\mathrm{A}} \otimes I \approx_{100\zeta_1^{1/4}} P_g^{\mathrm{A}} \otimes I, \qquad \texttt{(lem:orthonormalization-main-lemma)}$$
$$I \otimes G_g^{\mathrm{B}} \approx_{100\zeta_1^{1/4}} I \otimes P_g^{\mathrm{B}}. \qquad \texttt{(eq:G-self-consistency)}$$

Here P^{A} and P^{B} are projective submeasurements. The next step is to complete these submeasurements to projective measurements Q^{A} and Q^{B}. The value assigned there to the new error parameter is $\zeta_2 = 200\zeta_1^{1/4} + 40\zeta_1^{1/8}$, and the asserted estimate is

$$G_g^{\mathrm{A}} \otimes I \approx_{\zeta_2} Q_g^{\mathrm{A}} \otimes I,$$
$$I \otimes G_g^{\mathrm{B}} \approx_{\zeta_2} I \otimes Q_g^{\mathrm{B}}. \qquad \texttt{(eq:G-with-Q-A)}$$

3 The completion calculation

The needed completion result is the completion proposition in [JNV[+]20, Section *Preliminaries*]. [2] In the special case needed here, $A = \{A_a\}$ is a measurement, $B = \{B_a\}$ is a submeasurement, and $C = \{C_a\}$ is obtained from B by adding the missing mass $I - \sum_a B_a$ to one distinguished outcome. The proposition says that if

$$A_a \otimes I \approx_\delta B_a \otimes I$$

and the measurement A has self-consistency error ζ, then

$$A_a \otimes I \approx_{2\delta + 4\sqrt{\delta} + 2\zeta} C_a \otimes I.$$

For the projectivization step, A is the G-measurement, B is the projective submeasurement P, and C is the completed measurement Q. Thus $\delta = 100\zeta_1^{1/4}$, while the self-consistency parameter entering the completion proposition is $\zeta = \zeta_1$. Substitution gives

$$\begin{aligned} 2\delta + 4\sqrt{\delta} + 2\zeta &= 2(100\zeta_1^{1/4}) + 4\sqrt{100\zeta_1^{1/4}} + 2\zeta_1 \\ &= 200\zeta_1^{1/4} + 40\zeta_1^{1/8} + 2\zeta_1. \end{aligned}$$

[2] In the TeX source consulted here, this proposition occurs in `references/ldt-paper/preliminaries.tex`, lines 1101–1111, and has source label `prop:completing-to-measurement`.

Thus the scalar ζ_2 used in the completion estimate omits the final term $2\zeta_1$. This term is not a matter of notation: it is the contribution of the self-consistency error in the completion proposition.

4 Absorbing the missing term

In the non-vacuous branch of the cascade, the standing small-error hypotheses give $0 \le \zeta_1 \le 1$. In this range the omitted term is harmless at the scale already being used. Indeed $x \le x^{1/8}$ for $0 \le x \le 1$, and hence $2\zeta_1 \le 2\zeta_1^{1/8}$. The literal completion error is therefore bounded by $200\zeta_1^{1/4} + 42\zeta_1^{1/8}$.

Equivalently, the formal development keeps the exact composed error $2(100\zeta^{1/4}) + 4\sqrt{100\zeta^{1/4}} + 2\zeta$ and proves, for $0 \le \zeta \le 1$, the bound

$$2(100\zeta^{1/4}) + 4\sqrt{100\zeta^{1/4}} + 2\zeta \le 200\zeta^{1/4} + 42\zeta^{1/8}.$$

Applying this with $\zeta = \zeta_1$ gives the estimate needed for the completion step.[3]

Thus the scalar denoted ζ_2 in the corrected cascade used in the formal proof is $\zeta_2 = 200\zeta_1^{1/4} + 42\zeta_1^{1/8}$, rather than the scalar with coefficient 40 in the completion estimate.

5 Relation with the blueprint and Lean

The blueprint follows the corrected scalar [MIP26, Chapter *The main induction step*]. Its cascade definition uses $\zeta_2 = 200\zeta_1^{1/4} + 42\zeta_1^{1/8}$, and explains that the coefficient 42 absorbs the residual term $2\zeta_1$.[4] Later in the same chapter, the informal proof of the projectivization step again identifies the literal completion error as

$$2(100\zeta_1^{1/4}) + 4\sqrt{100\zeta_1^{1/4}} + 2\zeta_1.$$

Thus the blueprint agrees with the checked scalar calculation, rather than with the coefficient 40 in [JNV+20, Section *The main induction step*].

[3] The corresponding Lean declarations are `orthonormalizeAndCompleteError_le_absorbedZeta2`, `cascadeZeta2`, and `orthonormalizeAndCompleteError_zeta1_le_zeta2`. They are implemented in `MIPStarRE/LDT/MakingMeasurementsProjective/ProjectivizationChain/Basic.lean`, `MIPStarRE/LDT/Test/ErrorCascade/Definitions.lean`, and `MIPStarRE/LDT/Test/MainTheorem/ErrorScalars.lean`.

[4] In the blueprint source, this is in `blueprint/src/chapter/ch10_induction.tex`, lines 433–451, with source label `def:main-formal-error-cascade`.

The scalar correction is checked in the projectivization chain and in the final scalar cascade. These two parts of the development contain the literal completion error, the absorption into the coefficient 42, and the downstream cascade bounds.

The proof of the corrected final theorem is not part of this scalar calculation.[5] The calculation in issue #904 has been isolated and accounted for in the checked scalar and projectivization lemmas.

6 Conclusion

The verdict is that issue #904 records a real arithmetic omission in an intermediate error parameter, but not an obstruction arising from this scalar calculation to the final soundness estimate.

The formula for ζ_2 used in the completion estimate lacks the term $2\zeta_1$ arising from the completion proposition. The corrected scalar $200\zeta_1^{1/4} + 42\zeta_1^{1/8}$ absorbs this term in the regime $0 \leq \zeta_1 \leq 1$, and is still bounded by the error parameter appearing in the final soundness theorem. The discrepancy is therefore a quantitative error in an intermediate scalar, not by itself an obstruction to the final theorem.

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.

[MIP26] MIPStarRE Project. Blueprint for arxiv:2009.12982: Quantum soundness of the classical low individual degree test. `https://sirui-lu.com/MIPStarRE`, 2026. Lean blueprint for the MIPStarRE formalization.

[5] The later final-theorem repair treats the factor 400 in the large-k hypothesis and the nonzero-sampling condition $0 < k$ as documented statement corrections. Under those corrected hypotheses, the Section 6 small-error successor construction and the two-space final theorem are checked.