The evaluated-slice commutativity estimate for the processed-G measurements

August 24, 2026

1 The assertion

We examine the evaluated-slice commutativity lemma for the processed G measurements[1] in [JNV+20]. Under the three hypotheses of consistency with A, strong self-consistency, and boundedness, the G measurements approximately commute after evaluation for distinct point pairs. The conclusion is

$$\mathop{\mathbf{E}}_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{x},\boldsymbol{y}} \sum_{a,b} \left\| \left(G^{\boldsymbol{x}}_{[g(\boldsymbol{u})=a]} G^{\boldsymbol{y}}_{[h(\boldsymbol{v})=b]} - G^{\boldsymbol{y}}_{[h(\boldsymbol{v})=b]} G^{\boldsymbol{x}}_{[g(\boldsymbol{u})=a]} \right) \otimes I \left|\psi\right\rangle \right\|^2 \leq \nu, \qquad \nu = 48m\,(\sqrt{\gamma} + \sqrt{\zeta}). \tag{1}$$

2 The chain of approximations

The paper's proof expands the commutator square and isolates the $ABAB$ term. The path from $\mathbf{E}[\sum ABAB]$ to $\mathbf{E}[\sum ABA]$ passes through ten approximation steps:

1. Apply closenessOfIP with $A = G^{\boldsymbol{u},\boldsymbol{x}}_a \otimes I$ and $B = I \otimes A^{\boldsymbol{u},\boldsymbol{x}}_a$, at cost $2\sqrt{\zeta}$.
2. Swap $A^{\boldsymbol{v},\boldsymbol{y}}_b$ onto the right register, at cost $\sqrt{\zeta}$.[2]
3. Insert Bob's second measurement via closenessOfIP, at cost $2\sqrt{\zeta}$.
4. Apply point commutativity, at cost $6\sqrt{\gamma(m+1)}$.[3]
5. Remove the right-register $G^{\boldsymbol{x}}$ total, at cost $\sqrt{\zeta} + 6\sqrt{\gamma(m+1)}$.[4]
6. Reverse the closenessOfIP insertion of A on the left register, at cost $4\sqrt{\zeta}$ (steps 6–7 of the paper).

[1] The lemma carries the label lem:comm-data-processed-g in `references/ldt-paper/commutativity-G.tex`, lines 16–47, and is tracked in GitHub issue #760.

[2] Justified by the auxiliary claim clm:g-comm-stability in the cited paper.

[3] The point-commutativity theorem, labelled thm:commutativity-points.

[4] Justified by clm:g-comm-stability2.

7. Post-process G measurement outcomes, at cost $\sqrt{\zeta}$.[5]

8. Reverse the second-coordinate post-processing, at cost $\sqrt{\zeta}$.

The individual error terms sum to $12\sqrt{\zeta} + 12\sqrt{\gamma(m+1)}$. Multiplying by 2 (from the commutator-square expansion) and simplifying $12\sqrt{m+1} \leq 24m$ gives $\nu = 48m\,(\sqrt{\gamma} + \sqrt{\zeta})$.

3 The formal construction

The formal lemma replicates the same chain,[6] each paper step recorded as a named phase block. The final assembly adds these phases with the factor 2 from the commutator-square expansion, and the total matches the paper exactly.

The one substep that deserves explicit mention is the transition from the BAB term to the ABA term. In the paper this is justified by the projectivity of G, which makes

$$\mathop{\mathbf{E}}_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{x},\boldsymbol{y}} \sum_{a,b} G^{\boldsymbol{x}}_{[g(\boldsymbol{u})=a]} G^{\boldsymbol{y}}_{[h(\boldsymbol{v})=b]} = \mathop{\mathbf{E}}_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{x},\boldsymbol{y}} \sum_{a,b} G^{\boldsymbol{y}}_{[h(\boldsymbol{v})=b]} G^{\boldsymbol{x}}_{[g(\boldsymbol{u})=a]},$$

an exact equality. The formal proof uses the corresponding swap lemma,[7] which proves the average of the BAB term and the average of the ABA term equal as an *equality* (not an approximation), exactly matching the paper's step.

4 Conclusion

There is no mathematical divergence. The formal chain follows the paper's proof step-by-step, uses the same auxiliary lemmas (closeness of inner products, point commutativity, and G-commutativity stability), and achieves the same total error bound $48m\,(\sqrt{\gamma} + \sqrt{\zeta})$. The swap identity between the BAB and ABA averages is an exact equality in both the paper and the formalization. This note therefore documents a successful alignment, not a gap.

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.

[5] Justified by an auxiliary fact derived in the formalization.

[6] The lemma carries the name commDataProcessedG and is implemented in `MIPStarRE/LDT/Commutativity/ScalarApproximation/ProcessedG.lean`, lines 1972–2004. The error parameter, defined in `MIPStarRE/LDT/Commutativity/Scaffold/Core.lean` under the name commDataProcessedGError, evaluates to $48\,m\,(\gamma^{1/2} + \zeta^{1/2})$.

[7] Its declaration is evaluatedSliceCommutation_avg_swap_terms in `MIPStarRE/LDT/Commutativity/EvaluatedSliceCommutation/Averages.lean`, line 217.