The Zero-Sampling Boundary in the Main Formal Theorem

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1 At a glance

- **Difficulty: low as a mathematical obstruction, medium as a formal repair.** The scalar obstruction is immediate from the factor k^2 in the printed final error; deciding the replacement theorem form requires a source-statement judgment.
- **Estimated weight:** no new proof machinery is expected if the formal theorem is corrected to require $k > 0$. A fully formal counterexample would require a small explicit projective strategy with nonconstant point answers in degree zero.
- **Mathlib/project split:** approximately **5% Mathlib** and **95% project-local**. The only arithmetic is the simplification of the displayed error at $k = 0$; the issue concerns the LDT theorem statement.
- **Key Mathlib inputs:** elementary arithmetic on nonnegative real error parameters. The project-local inputs are the zero-k simplification of the final error and the degree-zero evaluation fact for polynomial-valued measurements.

2 Key theorem forms

Let σ be the projective strategy in the low individual degree test. Write m for the number of variables, q for the field size, d for the individual degree bound, k for the sampling parameter, and ϵ for the test-passing error, so that the strategy is assumed to pass the test with probability at least $1 - \epsilon$.

The final consistency error in the source theorem is denoted by ν. With this notation there are three relevant theorem forms.

1. **Printed source form.** The theorem `thm:main-formal` assumes $k \geq md$ and gives the three final consistency conclusions at
$$\nu = 100000k^2m^4\left(\epsilon^{1/40000} + (d/q)^{1/40000} + e^{-k/(2560000m^2)}\right).$$
2. **Current checked nonzero form.** The present formal route proves the corrected two-space source theorem under $k \geq 400md$ and $0 < k$. The role-register construction and the subsequent scalar absorption are checked inside the theorem, not supplied as external hypotheses.
3. **Zero-sampling boundary.** If $d = 0$ and $k = 0$, the printed hypothesis $k \geq md$ is satisfied, but the displayed error is exactly 0. The theorem then demands exact consistency with degree-zero polynomial measurements.

3 The source assertion

The source theorem in [JNV+20] states the final soundness theorem for an integer $k \geq md$.[1] This condition admits the boundary case $k = 0$ exactly when $d = 0$. At that boundary the printed error parameter is
$$100000 \cdot 0^2 \cdot m^4\left(\epsilon^{1/40000} + (0/q)^{1/40000} + e^{-0/(2560000m^2)}\right) = 0.$$
Thus the conclusion is no longer an approximate consistency statement. It is an exact consistency statement.

This is stronger than what the test-passing hypothesis implies. If ϵ is allowed to be large, the hypothesis that the strategy passes with probability at least $1 - \epsilon$ imposes no exact agreement condition on the point measurements. In degree zero, however, every polynomial outcome is a constant function on $\mathbb{F}_q^m$, so a polynomial-valued measurement evaluates to the same answer-valued measurement at every point. A strategy whose point measurement gives different deterministic answers at two different points can therefore satisfy the printed hypotheses for a sufficiently large ϵ, but cannot be exactly consistent with a single degree-zero polynomial measurement at both points.

[1] Tracked in GitHub issue #422. The relevant source passage is `references/ldt-paper/test_definition.tex`, lines 180–202.

For instance, take $m = 1$, $d = 0$, and $k = 0$, and let $q \geq 2$. On a one-dimensional classical Hilbert space, let the point measurement answer deterministically with value 0 at one field point and value 1 at another. With $\epsilon = 1$, the passing hypothesis is the vacuous inequality that the acceptance probability is at least 0. But any degree-zero polynomial measurement has a constant answer on $\mathbb{F}_q$, and so cannot agree exactly with both point measurements. Thus the zero value of ν would force a conclusion that this strategy does not satisfy.

This is not a defect of the final scalar cascade. The cascade simply records the printed factor k^2. At $k = 0$ that factor annihilates all error terms, including the term involving ϵ. Consequently the advertised conclusion is exact at $k = 0$ for every value of the test-passing parameter ϵ; the choice $\epsilon = 1$ in the example above is used only to make the printed hypothesis immediate. The obstruction itself is algebraic: a degree-zero polynomial measurement is constant on the field, whereas the point measurements in the example are not constant as functions of the queried point.

4 Comparison with the blueprint and formalization

The formalization now treats this boundary as a source-statement correction. The corrected Lean declaration MIPStarRE.LDT.Test.mainFormal_sourceStatement assumes $0 < k$, in addition to the separately confirmed large-k condition recorded in `docs/paper-gaps/issue-906-main-formal-k-bound.tex`. Under this corrected statement the source-boundary reduction and its small-error branch are proved by the two-space role-register route and the scalar-cascade absorption theorem.

The scalar equality mainFormalError_zero_k remains as the formal calculation explaining why the excluded boundary is singular: the advertised error is zero at $k = 0$. Thus the zero-sampling boundary is not another request for a package or bridge input; it is a statement-level correction.

5 Conclusion

The printed hypothesis $k \geq md$ is too weak at $d = 0, k = 0$ for the displayed final error. At that point the error parameter is exactly zero, while the test-passing hypothesis with arbitrary ϵ does not force exact agreement with a constant polynomial measurement. A source-faithful formal theorem must therefore either include a justified nonzero sampling condition, such as $0 < k$, or replace the zero-boundary conclusion by a statement whose error does not vanish at $k = 0$. The present formalization takes the first route: the corrected source theorem assumes $0 < k$, and the former zero-boundary obligation has been removed rather than discharged by further packaging.

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.