

Chapter 3 Same-Space Specializations of Two-Space Paper Propositions

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1 At a glance

- **Difficulty: medium as a cross-reference audit issue, low as a mathematical obstruction.** The faithful two-space forms are now proved `sorry`-free. The last formerly note-only entry, `consSubMeas`, was resolved by adding the two-space theorem `consSubMeas_heterogeneous`.
- **Estimated weight:** blueprint label changes for six propositions. The same-space forms remain available as auxiliary specializations for downstream uses that already work over $\iota \times \iota$.
- **Mathlib/project split:** 100% project-local. The issue concerns the chapter 3 blueprint `\lean{}` cross-references.
- **Key mathematical inputs:** the `*_heterogeneous` siblings in `MIPStarRE/LDT/Preliminaries/Triangles/Core.lean`, `.../Triangles/SimEq.lean`, `.../ComparisonCore.lean`, and `.../ComparisonProjective.lean`, together with the `consSubMeas_heterogeneous` theorem in `.../ConsistencyBridges.lean`. All relevant declarations are project-local; no external Mathlib inputs beyond the ambient linear-algebra and finite-dimensionality libraries.

2 Key theorem forms

The formalization isolates the following mathematical statements, all concerning the paper’s approximation relation $\simeq_\delta$ between submeasurements on two Hilbert spaces [JNV⁺20].¹

¹Tracked in GitHub issue #2338.

1. **Source theorem form.** For $A = \{A_a^x\}$ on $\mathcal{H}_A$ and $B = \{B_a^x\}$ on $\mathcal{H}_B$ and a state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the paper’s $\simeq_\delta$ relation is defined by

$$A_a^x \otimes I \simeq_\delta I \otimes B_a^x \quad \text{when} \quad \mathbf{E}_{x \sim \mathcal{D}} \sum_{a \neq b} \langle \psi | A_a^x \otimes B_b^x | \psi \rangle \leq \delta. \quad (\text{def:simeq})$$

The five propositions named below each assert a bound or implication involving this two-space relation (see `references/ldt-paper/preliminaries.tex`).

2. **Faithful formal form.** Each proposition is formalized as a `*_heterogeneous` declaration carrying independent index types ι_A, ι_B for the left and right submeasurement families and a state $|\psi\rangle \in \mathcal{H}_{\iota_A} \otimes \mathcal{H}_{\iota_B}$.² The same-space $(\iota \times \iota)$ specializations also exist and are retained for internal use, but the blueprint `\lean{}` links now target the two-space forms.
3. **Downstream theorem form.** The main-theorem path consumes the `*_heterogeneous` forms at `MIPStarRE/LDT/Test/MainTheorem/SourceRoleRegister/Final.lean` lines 449, 495, 505, 512, 563; `.../Completion.lean` line 925; and `.../Core.lean` line 312. No new hypotheses are needed beyond those already in the two-space forms.

The same-space forms are retained as auxiliary internal interfaces (they are mathematically correct as specializations and are occasionally convenient for proofs that already operate on a symmetrized state). However, the *faithful formalization* of each paper proposition is the `_heterogeneous` form, which is the version consumed on the main-theorem path.

3 Five retargeted links

The retargeting replaces five blueprint `\lean{}` cross-references. In each case the same-space specialization sets $\iota_A = \iota_B = \iota$ in the heterogeneous statement, reducing the two-space form

$$|\psi\rangle \in \mathcal{H}_{\iota_A} \otimes \mathcal{H}_{\iota_B}, \quad A_{\iota_A} \otimes I \simeq_\delta I \otimes B_{\iota_B}$$

to its diagonal

$$|\psi\rangle \in \mathcal{H}_\iota \otimes \mathcal{H}_\iota, \quad A_\iota \otimes I \simeq_\delta I \otimes B_\iota.$$

²The declarations are `triangleSub_heterogeneous`, `simeqTriangleInequality_heterogeneous`, `simeqToApprox_heterogeneous`, `approxToSimeq_heterogeneous`, and `simeqDataProcessing_heterogeneous` in `MIPStarRE/LDT/Preliminaries/`.

The heterogeneous form is the faithful transcription of `def:simeq` in [JNV⁺20, Section 3]; the same-space form is a correct specialization retained as an auxiliary interface.

3

The bounds match the paper exactly: $\delta + \sqrt{\epsilon}$ (triangle substitution), $\epsilon + 2\sqrt{\delta + \gamma}$ (triangle inequality), 2δ and δ (forward/converse), and δ (data processing).

4 consSubMeas_heterogeneous

The lemma `prop:cons-sub-meas` (`ch03\_preliminaries.tex:476`) is now linked to `consSubMeas_heterogeneous`, whose state space is $\iota_A \times \iota_B$. Its conclusion uses the two-space sandwich families `heterogeneousDiagonalSandwichFamily` and `heterogeneousTotalSandwichFamily`, corresponding respectively to $A_a^x \otimes B_a^x$ and $A^x \otimes B_a^x$.

The same-space theorem `consSubMeas` is retained as an auxiliary specialization. The only consumer of this specialization on the main-theorem path is `MainChain.lean:209`, which applies it at the symmetrized state `strategy.state : QuantumState ( $\iota \times \iota$ )`. The global proof never needs the fully general two-space form at that point. The retained same-space `consSubMeas` is therefore *downstream-sufficient*, while the blueprint now advertises the source-faithful two-space theorem.

5 Conclusion

- **Resolved (blueprint links retargeted):** all six two-space link retargetings listed above, including `consSubMeas_heterogeneous`.
- **Documented (same-space form retained as auxiliary):** `triangleSub`, `simeqTriangleInequality`, `simeqToApprox`, `approxToSimeq`, `simeqDataProcessing`.

³The five retargetings map `triangleSub` $\rightarrow$ `triangleSub_heterogeneous` (blueprint `prop:triangle-sub`, bound $\delta + \sqrt{\epsilon}$), `simeqTriangleInequality` $\rightarrow$ `simeqTriangleInequality_heterogeneous` (`prop:simeq-triangle-inequality`, bound $\epsilon + 2\sqrt{\delta + \gamma}$), `simeqToApprox` $\rightarrow$ `simeqToApprox_heterogeneous` (`prop:simeq-to-approx` forward, $\simeq_\delta \Rightarrow \approx_{2\delta}$), `approxToSimeq` $\rightarrow$ `approxToSimeq_heterogeneous` (converse, projective), and `simeqDataProcessing` $\rightarrow$ `simeqDataProcessing_heterogeneous` (`prop:simeq-data-processing`, bound δ). All five `_heterogeneous` forms are sorry-free. Four are consumed on the main-theorem path at `MIPStarRE/LDT/Test/MainTheorem/SourceRoleRegister/Final.lean` lines 449, 495, 505, 512, 563; `.../Completion.lean` line 925; and `.../Core.lean` line 312. `simeqDataProcessing_heterogeneous` is proved but not yet consumed on the main-theorem path.

`qDataProcessing`, and `consSubMeas` are retained as auxiliary internal interfaces. They are not removed or renamed. The `_heterogeneous` forms carry the main-theorem load.

There is no mathematical discrepancy. The retargeting makes the blueprint cross-references more faithful to the paper’s two-space definition of $\simeq_\delta$.⁴

References

- [JNV⁺20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.

⁴Related: GitHub issue #2337 (same-space cleanup of the `mainFormal` interface, the Preliminaries-layer analogue). The same-space forms here are distinct from the genuine open mathematical gap in GitHub issue #1093 (the §9 η / total-overlap term).