Issue 1622: The Degree-Zero Branch of Low-Degree Pasting

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1 At a glance

- **Status: discharged.** The degree-zero branch is now proved as MIPStarRE.LDT.Pasting.ldPastingDegreeZeroBranch, and the unrestricted source-facing theorem MIPStarRE.LDT.Pasting.ldPasting assembles this branch with the nontrivial, large-error, and $k = 0$ branches. The axiom audit checks both the degree-zero branch and the unrestricted theorem with standard Lean axioms only.
- **Difficulty.** Medium. The obstruction is not a new analytic estimate, but the positive-degree proof uses a scalar absorption which is no longer available when $d = 0$. This was the obstruction before the degree-zero proof was supplied.
- **Estimated weight.** One branch theorem, probably 100–300 lines of formal proof, depending on how directly the existing slice-consistency and point-consistency interfaces can be reused.
- **Mathlib/project split.** The expected proof is mostly project-local. Mathlib should supply only elementary facts about finite sums, natural-number inequalities, and monotonicity of real powers.
- **Key mathematical inputs.** Degree-zero polynomials are constant; the pasted slices satisfy the source completeness and consistency hypotheses; and the final measurement must satisfy the same point-consistency conclusion as in the unrestricted theorem.

2 Key theorem forms

We use the notation of the low-degree pasting theorem. Let S be an $(\varepsilon, \delta, \gamma)$-good symmetric strategy for the $(m + 1, q, d)$ low individual degree test. For each $x \in \mathbb{F}_q$, let $G^x \in \mathrm{PolySub}(m, q, d)$ be the projective submeasurement on the x-slice, and assume the completeness, consistency, strong self-consistency, and boundedness hypotheses stated in the source theorem.

1. **Source theorem form.** If $k \geq 400md$, then there is a pasted measurement $H \in \mathrm{PolyMeas}(m+1, q, d)$ whose point-consistency error is

$$\sigma = \kappa \left(1 + \frac{1}{100m}\right) + 2\nu + \exp\left(-\frac{k}{80000m^2}\right),$$

where

$$\nu = 100k^2 m \left(\varepsilon^{1/32} + \delta^{1/32} + \gamma^{1/32} + \zeta^{1/32} + (d/q)^{1/32}\right).$$

This is the theorem form in the paper; it does not contain the hypothesis $0 < d$.

2. **Restricted construction theorem form.** One formal construction theorem constructs such an H in the nontrivial regime

$$\gamma \leq 1, \qquad \zeta \leq 1, \qquad d \leq q, \qquad 0 < d, \qquad 1 \leq k.$$

This is useful proof content, but it is a restricted construction theorem, not the source theorem.

3. **Degree-zero branch form.** The complementary branch is the same conclusion under the source hypotheses together with

$$d = 0, \qquad 1 \leq k.$$

This branch has now been proved, restoring the unrestricted theorem without changing its public hypotheses.[1]

3 The source assertion

This note concerns the low-degree pasting theorem in [JNV+20].[2] After stating the theorem, the paper says that the proof may assume $\varepsilon, \delta, \gamma, \zeta, d/q \leq 1$, since the desired bound is trivial if one of these quantities is at least 1. This is a reduction to the small-error regime. It is not a hypothesis $0 < d$, and it does not by itself dispose of the case $d = 0$.

The point is visible in the error term itself. When $d = 0$, the displayed contribution $(d/q)^{1/32}$ is zero. Thus any proof step that pays a positive loss of order $1/q$ or k^2/q cannot be justified merely by appealing to the term $(d/q)^{1/32}$.

[1] The source-facing Lean theorem is MIPStarRE.LDT.Pasting.ldPasting. The restricted construction is MIPStarRE.LDT.Pasting.ldPastingNontrivial. The degree-zero branch is MIPStarRE.LDT.Pasting.ldPastingDegreeZeroBranch in `MIPStarRE/LDT/Pasting/Bernoulli/Final.lean`. The repair is tracked by GitHub issue #1622 and documented in PR #1633.

[2] The TeX source used here is `references/ldt-paper/ld-pasting.tex`, lines 12–55. The theorem is labelled `thm:ld-pasting`.

4 Why the positive-degree proof does not specialize

The nontrivial proof follows the sampled distinct-tuple construction in the paper. In the estimates corresponding to the H_B-consistency and overall-outcome steps, the argument passes between independently sampled heights and distinct sampled heights. This passage introduces a distinctness loss of order k^2/q.

For $0 < d$, this loss is absorbed into the displayed error term by the scalar chain

$$\frac{1}{q} \leq \frac{d}{q} \leq \left(\frac{d}{q}\right)^{1/32},$$

in the small-error regime. The first inequality is exactly where the positive-degree hypothesis enters. When $d = 0$, the right-hand side is zero, so the same scalar argument cannot close the estimate.

This is not evidence that the paper theorem should acquire the hypothesis $0 < d$. Rather, it shows that the formal proof has separated two branches: the existing positive-degree argument and the remaining degree-zero argument. In the degree-zero branch, the formal proof uses that every degree-zero polynomial is constant, and obtains point consistency from the vertical line-test and slice-consistency hypotheses without paying a distinct-tuple loss that must be absorbed by d/q.

5 Comparison with the blueprint and formalization

The formal development keeps the unrestricted source-facing theorem form visible. Its proof dispatches the large-γ, large-ζ, large-d/q, $k = 0$, positive-degree nontrivial, and degree-zero cases. The degree-zero branch is proved by MIPStarRE.LDT.Pasting.ldPastingDegreeZeroBranch, not assumed as an additional public hypothesis of the theorem.

This is the intended paper-realignment boundary. The restricted positive-degree theorem remains useful proof content, but it should be cited only as a nontrivial-regime construction theorem. It should not replace the source theorem in the blueprint, because its public hypotheses are stronger than the cited statement.

6 Conclusion

There is no justification for adding $0 < d$ to the paper-facing low-degree pasting theorem. The theorem-level statement remains the unrestricted statement of the source. The formerly missing degree-zero branch is now proved: it constructs the pasted measurement and point-consistency estimate from $d = 0$, $1 \leq k$, and the source hypotheses, and the unrestricted theorem assembles it with the nontrivial and large-error branches.

References

[JNV$^+$20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.