Issue 1230: How the Self-Improvement SDP Is Used

2026-05-05

1 At a glance

- **Status: discharged in PR #1708.** The former issue was the full **strong-duality and complementary-slackness producer** for the self-improvement SDP, not just the already-fixed submeasurement interface. The current formalization proves this producer in the canonical finite- dimensional SDP model and transports it to the paper-facing statement.
- **Difficulty: medium.** The underlying SDP argument is standard finite-dimensional convex duality, but the formalization must choose a canonical block SDP whose optimal primal witness has no slack block without assuming the auxiliary dominance condition $I \preceq Z$.
- **Estimated weight: medium.** The completed proof uses a substantial project-local canonical SDP layer, while the analytic inputs are ordinary finite-dimensional matrix-order and separation facts.
- **Traceability.** The issue is tracked in GitHub issue #1230; the canonical strong-duality route was completed in PR #1708, the obsolete dominance wrappers were removed in PR #2422, and the reduced `SdpStatement`/`sdp` interface was retired in PR #2430.
- **Implemented route:** project-local canonicalization, compactness and separation for the canonical SDP image cone, equality of primal and dual optima, complementary slackness from weak duality, saturation of the auxiliary slack block, and the self-improvement adapter.
- **Mathlib/project split:** roughly **35% Mathlib** and **65% project-local**. Mathlib supplies matrix order, traces, positive-semidefinite cones, convex cones, and separation tools; the project must build the paper-specific block SDP and helper interface.

- **Key Mathlib inputs:** positive semidefinite matrices, trace pairing, convex cones and dual cones, finite-dimensional linear maps, and hyperplane separation for closed/proper cones.[1]

2 Key theorem forms to develop

The following theorem forms are the ones needed for the LDT self-improvement argument and are now represented in Lean. They are deliberately stronger than the local issue 196 submeasurement fix.

1. **PSD effect bounds.** For every polynomial g, prove
$$0 \preceq A_g \preceq I, \qquad A_g = \mathop{\mathbf{E}}_{\boldsymbol{u}} A^{\boldsymbol{u}}_{g(\boldsymbol{u})}.$$
This supplies dual feasibility of the strict witness $Z = 2I$ and nonnegative objective gain when slack mass is completed.

2. **Canonical block SDP equivalence.** The paper primal over submeasurements is represented by the canonical PSD variable $X = \mathrm{diag}(T_{g_1}, \dots, T_{g_M}, S)$ with
$$\sum_g T_g + S = I, \qquad \langle C, X\rangle = \sum_g \mathrm{tr}(T_g A_g).$$
Its dual is the paper dual $Z \succeq A_g$.

3. **Strong duality.** The formal proof uses the strict witnesses $T_g = (2M)^{-1} I$ and $Z = 2I$ for feasibility and proves finite-dimensional SDP zero gap by separating the canonical primal image cone: there are optimal X_* and Z_* with equal primal and dual values.

4. **Saturated slackness witness.** Choose an optimal primal witness with **no slack block**, i.e. $\sum_g T_g = I$, and a dual optimum Z such that
$$Z \succeq A_g, \qquad T_g(Z - A_g) = 0 \quad \text{for all } g.$$
Saturation is obtained by completing primal slack into an outcome whose objective coefficient is positive semidefinite, without decreasing the objective. Slackness is then recovered from zero duality gap and positive-semidefinite dual slacks. This route avoids the stronger and generally unavailable hypothesis $I \preceq Z$.

[1] Representative APIs live around `Mathlib.LinearAlgebra.Matrix.PosDef`, `Mathlib.Analysis.Matrix.Order`, `Mathlib.Analysis.Convex.Cone.Dual`, `Mathlib.Analysis.Convex.Cone.InnerDual`, and `Mathlib.Geometry.Convex.Cone.*`.

5. **Self-improvement SDP output.** Package the previous item as the exact data consumed by the helper proof:

$$T \text{ is a measurement}, \qquad Z \succeq 0, \qquad Z \succeq A_g, \qquad T_g Z = T_g A_g.$$

The dominance-carrying interfaces that ask for the extra field $I \preceq Z$ are auxiliary conditional routes, not consequences of the SDP dual constraints and not hypotheses of the paper-facing theorem. The faithful paper route uses the saturated-witness argument above.[2]

3 The source SDP

The self-improvement lemma in [JNV+20] uses one SDP.[3] For $A_g = \mathbf{E}_{\boldsymbol{u}} A^{\boldsymbol{u}}_{g(\boldsymbol{u})}$, the primal and dual are

$$\sup_{\{T_g\}} \quad \sum_g \operatorname{tr}(T_g A_g), \quad T_g \succeq 0, \quad \sum_g T_g \preceq I, \qquad (\texttt{eq:primal-objective})$$
$$\inf_Z \quad \operatorname{tr}(Z), \qquad Z \succeq A_g \qquad \text{for every } g. \quad (\texttt{eq:dual-objective})$$

The primal object is a **submeasurement**. The SDP lemma then selects an optimal pair with

$$\sum_g T_g = I, \qquad T_g Z = T_g A_g. \qquad (\texttt{lem:sdp}, \texttt{eq:slater})$$

4 The dominance condition is auxiliary

The dual constraint $Z \succeq A_g$ does not imply $Z \succeq I$. In the scalar one-outcome case with $A_g = \frac{1}{2}I$, the trace-minimizing dual solution is $Z = \frac{1}{2}I$, which is feasible and optimal but does not dominate the identity. Thus the implication

$$S \succeq 0, \qquad SZ = 0, \qquad Z \succeq A_g \ \forall g \quad \Longrightarrow \quad S = 0$$

is not a valid consequence of dual feasibility alone. It becomes valid after adding the stronger hypothesis $I \preceq Z$, because then $Z \succeq 0$ and $S(Z-I) \succeq 0$

[2] The current bridge isolating this interface issue is `MIPStarRE/LDT/SelfImprovement/Theorems/Results/SdpMatrixBridge.lean`.

[3] The SDP is in `references/ldt-paper/self_improvement.tex`, lines 62–88. The strict-feasibility and complementary-slackness paragraph is in lines 168–190.

after commuting the positive factors; from $SZ = 0$ this gives $0 \preceq -S$, hence $S = 0$.

The corresponding matrix-order assertion is therefore deliberately conditional: it is valid under the additional hypothesis $I \preceq Z$, but not under dual feasibility alone. Earlier versions of the formalization packaged the resulting SDP data in declarations whose names ended in WithDominance; those wrappers have now been removed from the live SDP interface. Their former existence did not prove that an arbitrary optimal dual witness selected by Slater duality satisfies $I \preceq Z$.

5 How the proof uses the optimum

The paper defines

$$H_h^u = A_{h(u)}^u T_h A_{h(u)}^u, \qquad H_h = \mathop{\mathbf{E}}_{\boldsymbol{u}} H_h^{\boldsymbol{u}}.$$

(`self_improvement.tex:203--220`)

The proof uses **exactly three SDP outputs**: submeasurement bounds make H^u and H submeasurements; saturation collapses sums over T_g; and slackness replaces A_g by Z after left multiplication by T_g. For example,

$$\mathop{\mathbf{E}}_{\boldsymbol{u}} \sum_h \langle\psi| \left(T_h A_{h(\boldsymbol{u})}^{\boldsymbol{u}}\right) \otimes I \,|\psi\rangle = \sum_h \langle\psi| \left(T_h Z\right) \otimes I \,|\psi\rangle .$$

Dual feasibility $Z \succeq A_g$ then supplies the boundedness certificate carried through the induction.[4]

6 Blueprint and formalization

The blueprint matches the paper: submeasurement primal, strict feasible witness, then a saturated slackness-producing optimum.[5] Lean keeps the feasible primal boundary explicit as a submeasurement, while the selected saturated optimum is exposed to the slackness-carrying statement as a measurement. The construction supplied by PR #1708 is the **producer** for this stronger statement.[6]

[4] The two slackness substitutions are in `references/ldt-paper/self_improvement.tex`, lines 397–400 and 580–584. The boundedness conclusion is restated in `references/ldt-paper/inductive_step.tex`, lines 277–284, 315–322, 477–484, and 542–549.

[5] See `blueprint/src/chapter/ch07_self_improvement.tex`, especially `lem:sdp-uniform-feasible-witness`, `lem:sdp-matrix-feasible-bounds`, `lem:sdp-matrix-slackness-output`, and `lem:sdp`.

[6] Representative declarations are sdpStrictPrimalSubMeas, sdpPrimalObjective, sdpComplementarySlacknessEquation, and averagedSandwichedPolynomialSubMeas in `MIPStarRE/`

7 Conclusion

There is a genuine distinction between the unconditional SDP statement used in the paper and the former conditional dominance bridges. Those former wrappers proved correct conditional statements: if a canonical optimal pair is supplied with $I \preceq Z$, then complementary slackness forces the slack block to vanish and the pair yields a complete measurement satisfying the displayed slackness equations. This does not by itself prove the unconditional conclusion of `lem:sdp`; the unconditional conclusion is instead proved by the canonical strong-duality and saturation route.

The faithful completion of the paper route is now represented by MIPStarRE.LDT.SelfImprovement.matrixSdpCanonicalStrongDuality and MIPStarRE.LDT.SelfImprovement.sdp_statement_with_slackness: the formalization proves strong duality with complementary slackness for the canonical block SDP, produces a saturated optimal witness without assuming $I \preceq Z$, and connects that witness to the self-improvement interface. The reduced theorem sdp, its SdpStatement record, and the wrapper selfImprovementHelperConstruction have been removed; the remaining paper-facing route is the slackness-carrying statement SdpStatementWithSlackness.

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.

`LDT/SelfImprovement/Defs.lean`. The active source-facing interface is SdpStatementWithSlackness in `MIPStarRE/LDT/SelfImprovement/Theorems/Statements.lean`; the former reduced SdpStatement interface was retired in PR #2430.