# Issue 1100: Endpoint Branches in the Proof of `lem:projective-non-measurement`

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## 1 At a glance

- **Difficulty.** Low. The paper proof treats the nontrivial small-error regime explicitly, while the endpoint $\zeta = 0$ and the large-error regime require separate formal branches.
- **Estimated weight.** A local spectral endpoint argument together with a trivial large-error construction; both are already represented in the current Lean development.
- **Mathlib/project split.** Mostly project-local. Mathlib supplies the spectral calculus and matrix-order facts, while the project supplies the measurement, state-dependent distance, and spectral-truncation interfaces.
- **Key mathematical inputs.** The source almost-projectivity estimate `eq:A-looks-projective`, the small-error truncation theorem, the $\zeta = 0$ spectral projection argument, and the large-error fallback.

## 2 Key theorem forms

The formalization separates the local theorem used in the paper from the unconditional theorem exposed to later Lean code.

1. **Small-error source branch.** Under the hypotheses of `eq:A-looks-projective` and the standing assumption $0 < \zeta \leq 1/4$, construct projectors $R_a$ satisfying the state-dependent-distance estimate and $\sum_a R_a \leq (1 + 2\sqrt{\zeta})I$.

2. **Endpoint branch.** When $\zeta = 0$, replace the positive truncation threshold by the spectral projector onto the 1-eigenspace and prove the exact state-dependent-distance conclusion.

3. **Unconditional packaging theorem.** Combine the small-error branch, the endpoint branch, and the trivial large-error branch into the standalone projective-non-measurement witness consumed by the orthonormalization construction.[1]

# 3 The cited statement

This note concerns the lemma "Rounding to projectors" in [JNV+20, Section 5].[2] The cited lemma states that, for a measurement $A = \{A_a\}$, there exists a family of projectors $\{R_a\}$ such that

$$A_a \otimes I \approx_{2\sqrt{\zeta}} R_a \otimes I$$

and, writing $R = \sum_a R_a$,

$$R \le (1 + 2\sqrt{\zeta})I.$$

In the surrounding proof, the parameter $\zeta$ is the error term already fixed in the orthonormalization-main-lemma argument.

Just before this lemma, the paper derives

$$\sum_a \langle\psi| (A_a - A_a^2) \otimes I |\psi\rangle \le 2\zeta. \qquad \texttt{(eq:A-looks-projective)}$$

This is the only input used in the proof of the rounding lemma: once `eq:A-looks-projective` has been obtained, the auxiliary measurement $B$ no longer appears.

[1] The relevant declarations are projectiveNonMeasurement_of_sourceAlmostProjective_two_mul, projectiveNonMeasurement_of_sourceAlmostProjective_two_mul_full, projectiveNonMeasurement_of_sourceAlmostProjective_full, and spectralTruncationStatement_of_sourceAlmostProjective in `MIPStarRE/LDT/MakingMeasurementsProjective/SpectralTruncation/ProjectiveNonMeasurement.lean`.

[2] Tracked in GitHub issue #1100. In the TeX source consulted here, the statement and proof are in `references/ldt-paper/orthonormalization.tex`, lines 414–531.

## 4 The large-error branch

The paper does handle the large-$\zeta$ region, but it does so one level above the rounding lemma. At the start of the proof of the orthonormalization-main lemma it says that the argument is trivial when $\zeta > 1/4$, and therefore continues under the standing assumption

$$\zeta \leq 1/4. \qquad (\texttt{eq:assumption-on-zeta})$$

This reduction appears in `references/ldt-paper/orthonormalization.tex`, lines 375–379, before the derivation of `eq:A-looks-projective` and before the proof of `lem:projective-non-measurement`. Thus the cited argument for the rounding lemma is intended to be read only in the small-error regime.

For the paper as written, this is enough: the lemma is invoked only inside the proof which has already restricted to $\zeta \leq 1/4$. A standalone formal declaration, however, cannot rely on that surrounding reduction unless it is built into the declaration itself. If one wants a public theorem whose conclusion is the rounding witness for all $\zeta \geq 0$, then the trivial large-error branch must be supplied explicitly in the formal development even though the paper leaves it implicit at the level of `lem:projective-non-measurement`.

## 5 The endpoint $\zeta = 0$

The proof of the rounding lemma introduces an auxiliary threshold parameter $\delta$ satisfying

$$0 < \delta \leq 1/2. \qquad (\texttt{eq:bound-on-delta})$$

The positivity of $\delta$ is used throughout the truncation argument, because the scalar inequality

$$(x - \mathsf{trunc}_\delta(x))^2 \leq \frac{1}{\delta}(x - x^2)$$

and its operator consequence

$$(A_a - R_a)^2 \leq \frac{1}{\delta}(A_a - A_a^2)$$

both divide by $\delta$. The same factor $1/\delta$ is then carried into the state-dependent-distance estimate in lines 487–493 of the source file.

At the end of the proof, the paper sets

$$\delta = \sqrt{\zeta}.$$

This substitution appears in `references/ldt-paper/orthonormalization.tex`, lines 528–529. Together with `eq:bound-on-delta`, it covers exactly the regime

$$0 < \zeta \le 1/4.$$

When $\zeta = 0$, the statement of the lemma still makes sense, but the printed proof no longer applies literally, because it would require $\delta = 0$ while simultaneously using $1/\delta$.

This is therefore a proof-level endpoint omission rather than a theorem-level counterexample. The mathematical content of the $\zeta = 0$ case is clear. If

$$\sum_a \langle\psi| (A_a - A_a^2) \otimes I |\psi\rangle = 0,$$

then each summand is a nonnegative expectation and hence vanishes. Writing $A_a = \sum_i \lambda_{a,i} |u_{a,i}\rangle \langle u_{a,i}|$ with $0 \le \lambda_{a,i} \le 1$, one may define $R_a$ to be the spectral projector onto the 1-eigenspace of $A_a$. Then $R_a \le A_a$, so

$$\sum_a R_a \le \sum_a A_a = I,$$

which is stronger than the bound claimed in the lemma. Moreover the vanishing of $\langle\psi| (A_a - A_a^2) \otimes I |\psi\rangle$ forces the state to have no weight on eigenspaces with eigenvalue strictly between 0 and 1, so

$$A_a \otimes I \approx_0 R_a \otimes I.$$

The endpoint is therefore valid, but it requires a separate spectral argument.

# 6 Relation with the formal development

The current Lean declarations make this distinction explicit.[3] Thus the present formal development has discharged the unconditional spectral-

[3] The constructive small-error branch with the paper's $2\zeta$ source-defect bound is formalized by projectiveNonMeasurement_of_sourceAlmostProjective_two_mul in `MIPStarRE/LDT/MakingMeasurementsProjective/SpectralTruncation/ProjectiveNonMeasurement.lean`. The unconditional spectral-truncation producers projectiveNonMeasurement_of_sourceAlmostProjective_two_mul_full and projectiveNonMeasurement_of_sourceAlmostProjective_full combine that branch with the exact $\zeta = 0$ endpoint and the trivial large-error case, and spectralTruncationStatement_of_sourceAlmostProjective records the result directly as the spectral-truncation statement used by the later Section 5 construction.

truncation packaging. The small-error proof still corresponds to the part of the cited argument in which one may set $\delta = \sqrt{\zeta}$ and satisfy `eq:bound-on-delta`; the zero and large-error regimes are supplied by separate formal branches. The remaining Section 5 work lies in the later locality-preserving repair and completion arguments, not in the endpoint packaging isolated in this note.

In other words, the formal development has not discovered a contradiction in the paper statement. It has isolated the exact places where the cited proof uses positivity of $\delta$, and therefore where the proof must be supplemented if one wants a theorem stated without side conditions.

## 7 Conclusion

The paper does not need a separate large-$\zeta$ calculation inside the proof of `lem:projective-non-measurement`, because that regime has already been removed in the surrounding proof of the orthonormalization-main lemma. The formal development must nevertheless add such a branch if it exposes the rounding lemma as an unconditional standalone theorem.

The more delicate omission is the endpoint $\zeta = 0$. The statement of the rounding lemma remains meaningful there, but the printed proof covers only $0 < \zeta \leq 1/4$, because it introduces a positive parameter $\delta$ and later sets $\delta = \sqrt{\zeta}$. Thus the cited source leaves the endpoint argument implicit. A faithful unconditional formalization should keep the paper's nontrivial branch for $0 < \zeta \leq 1/4$, and add separate proofs for the large-error region and for the exact endpoint $\zeta = 0$.

## References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.