Issue 1099: The Line-169 Replacement Step in the Main-Induction Projectivization Chain

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1 At a glance

- **Difficulty.** This is a local quantitative repair in the projectivization part of the main-induction proof. The mathematical obstruction is not the existence of the completed measurement, but the size of the error term after replacing an orthonormalized submeasurement by its completion.
- **Estimated weight.** The remaining formal work is small to medium: the local comparison lemmas are checked, while later uses must continue to thread the corrected error term rather than the printed line-169 error.
- **Mathlib/project split.** Mathlib supplies the Hilbert-space algebra, positivity, and Cauchy–Schwarz estimates. The project supplies the state-dependent distance relations, completion construction, and projectivization-chain bookkeeping.
- **Key mathematical inputs.** The checked inputs are `prop:triangle-sub`, the easy approximation-to-match-mass comparison, and monotonicity of diagonal match mass under canonical completion of a projective submeasurement.

2 Key theorem forms

The paper's printed line-169 replacement would preserve the error ζ_1. The formalized substitution theorem instead gives the generic bound

$$Q_g^{\mathrm{A}} \otimes I \simeq_{\zeta_1+\sqrt{\zeta_2}} I \otimes G_g^{\mathrm{B}}.$$

The checked local repair uses the pre-completion orthonormalization estimate and the monotonicity of canonical completion to obtain the sharper corrected bound

$$Q^{\mathrm{A}}_g \otimes I \simeq_{\zeta_1+10\zeta_1^{1/8}} I \otimes G^{\mathrm{B}}_g.$$

This bound is the one recorded by the projectivization-chain repair lemmas.

3 The cited step

This note concerns the projectivization step in the proof of the main induction theorem in [JNV+20].[1] At this point in the argument, the paper has already produced polynomial-outcome measurements G^{A} and G^{B} with consistency error ζ_1, projective submeasurements P^{A} and P^{B} from orthonormalization, and completed projective measurements Q^{A} and Q^{B}.

The paper first states the pre-projective consistency estimate

$$G^{\mathrm{A}}_g \otimes I \simeq_{\zeta_1} I \otimes G^{\mathrm{B}}_g, \qquad (\texttt{eq:G-self-consistency})$$

then the completion estimate

$$G^{\mathrm{A}}_g \otimes I \approx_{\zeta_2} Q^{\mathrm{A}}_g \otimes I, \qquad (\texttt{eq:G-with-Q-A})$$

and finally says that Proposition `prop:triangle-sub`, applied to these two relations, implies

$$Q^{\mathrm{A}}_g \otimes I \simeq_{\zeta_1} I \otimes G^{\mathrm{B}}_g. \qquad (\text{paper line 169})$$

The same claim is then used, after data processing, in the point-evaluation transport.

4 Why the printed inference is incorrect

The substitution proposition `prop:triangle-sub` does not preserve the original consistency error when one replaces a measurement by an $\approx_\epsilon$-close one. Its checked content is that from

$$A \simeq_\delta C \qquad \text{and} \qquad A \approx_\epsilon B$$

[1] Tracked in GitHub issue #1099. In the TeX source consulted here, the relevant passage is in `references/ldt-paper/inductive_step.tex`, lines 150–180, and the specific replacement step is lines 167–173.

one obtains

$$B \simeq_{\delta+\sqrt{\epsilon}} C.$$

Thus applying it directly to `eq:G-self-consistency` and `eq:G-with-Q-A` yields only

$$Q_g^{\mathrm{A}} \otimes I \simeq_{\zeta_1+\sqrt{\zeta_2}} I \otimes G_g^{\mathrm{B}},$$

not the printed exact ζ_1 bound.

This is not a formalization artefact. It is the quantitative content of the cited proposition itself. The paper's displayed use of `prop:triangle-sub` therefore drops a square-root term.

5 A more local corrected estimate

There is, however, a sharper local repair than the naive $\zeta_1 + \sqrt{\zeta_2}$ route. One should compare with the orthonormalized submeasurement *before* completion.

Orthogonalization gives

$$G_g^{\mathrm{A}} \otimes I \approx_{100\zeta_1^{1/4}} P_g^{\mathrm{A}} \otimes I.$$

For a fixed partner measurement G^{B}, Proposition `prop:easy-approx-from-approx-delta` bounds the change in diagonal match mass by the square root of the $\approx$-error. Specializing that proposition to the single-question setting gives

$$\Big|\sum_g \langle \psi, (G_g^{\mathrm{A}} \otimes G_g^{\mathrm{B}})\psi\rangle - \sum_g \langle \psi, (P_g^{\mathrm{A}} \otimes G_g^{\mathrm{B}})\psi\rangle\Big| \le \sqrt{100\zeta_1^{1/4}} = 10\zeta_1^{1/8}.$$

Since `eq:G-self-consistency` says that the first match mass is at least $1-\zeta_1$, one obtains

$$\sum_g \langle \psi, (P_g^{\mathrm{A}} \otimes G_g^{\mathrm{B}})\psi\rangle \ge 1 - \zeta_1 - 10\zeta_1^{1/8}.$$

Now complete P^{A} at a distinguished outcome to obtain Q^{A}. The completion adds the positive residual $I - \sum_g P_g^{\mathrm{A}}$ to one outcome, so the diagonal match mass against the fixed partner measurement G^{B} can only increase. Therefore

$$\sum_g \langle \psi, (Q_g^{\mathrm{A}} \otimes G_g^{\mathrm{B}})\psi\rangle \ge 1 - \zeta_1 - 10\zeta_1^{1/8},$$

which is equivalent to the corrected consistency estimate

$$Q_g^{\mathrm{A}} \otimes I \simeq_{\zeta_1 + 10\zeta_1^{1/8}} I \otimes G_g^{\mathrm{B}}.$$

The Bob-side mirror is identical after exchanging the tensor factors.

This repaired bound is strictly local to the projectivization step. It uses only the already-derived G–P orthonormalization comparison and the positivity of canonical completion. Quantitatively it is much better than the direct completed-measurement substitution route $\zeta_1 + \sqrt{\zeta_2}$, because it inserts the square root before the completion scalar has been expanded.

6 Relation with the blueprint and Lean

The blueprint remark on the Lean line-169 interface now records the active transport routes and the auxiliary completion fact they use:[2] the generic $\zeta_1 + \sqrt{\zeta_2}$ substitution route, the sharper local repaired route $\zeta_1 + 10\zeta_1^{1/8}$ described above, and the checked completion helper `ProjectivizationMatchMassMonotonicity.completeAtOutcomeProj_left_matchMass_ge` showing that canonical completion adds no further diagonal match-mass loss.

The formal development proves the repaired route by checked declarations.[3] These declarations do not prove the paper's exact ζ_1 line. They prove the local corrected error $\zeta_1 + 10\zeta_1^{1/8}$.

The generic corrected route $\zeta_1 + \sqrt{\zeta_2}$ remains available as the literal consequence of `prop:triangle-sub`, but propagating that loss all the way through the final point-transport triangle would force a larger final envelope at the $1/65536$ scale. The present formal base-case assembly instead uses the sharper local pre-completion repair, so the public final envelope keeps the older $1/40000$ scale.

The former exact-ζ_1 formal interface has now been retired from the active Lean development. The formal development instead cleanly distinguishes

[2]See `blueprint/src/chapter/ch04_projective.tex`, the remark `rem:lean-line169-projectivization-match-mass`.

[3]The generic pre-completion transport theorems are MIPStarRE.LDT.MakingMeasurementsProjective.ProjectivizationLine169Repair.leftConsistency_of_completion_and_sdd and MIPStarRE.LDT.MakingMeasurementsProjective.ProjectivizationLine169Repair.rightConsistency_of_completion_and_sdd in `MIPStarRE/LDT/MakingMeasurementsProjective/ProjectivizationChain/Line169Repair.lean`. Specializing the $\approx$-error to MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalizationError gives MIPStarRE.LDT.MakingMeasurementsProjective.ProjectivizationLine169Repair.leftConsistency_with_orthonormalization_loss and its Bob-side mirror.

two questions:

- the paper's printed use of `prop:triangle-sub`, which is incorrect as stated and has the checked local repair above; and
- the stronger exact-ζ_1 replacement step, which would require an additional construction-specific monotonicity theorem not supplied by the cited proof and is no longer represented by a live Lean interface.

7 Conclusion

The verdict is that the paper's line-169 replacement step contains a genuine quantitative error. Applying `prop:triangle-sub` to the completed measurement Q^{A} does not yield the printed exact ζ_1 bound.

The most local corrected estimate presently available is

$$Q_g^{\mathrm{A}} \otimes I \simeq_{\zeta_1 + 10\zeta_1^{1/8}} I \otimes G_g^{\mathrm{B}},$$

with the analogous Bob-side statement. This repair is already formalized and is substantially sharper than the naive $\zeta_1 + \sqrt{\zeta_2}$ route, but it is still weaker than the paper's exact line. The formal public theorem keeps the older $1/40000$ envelope by using this sharper local repair rather than the generic corrected route. Whether the exact ζ_1 transport can be recovered from a more refined construction-level argument remains a separate open question.

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.