# Issue 1093: The Total-Overlap Term in Projective Point-Consistency Transport

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## 1 At a glance

- **Difficulty.** Medium. The issue is a missing submeasurement-total term in a triangle-substitution step, rather than a failure of the underlying state-dependent distance estimate.
- **Estimated weight.** One project-local transport estimate and its scalar absorption into the self-improvement final-fields construction.
- **Mathlib/project split.** The obstruction is project-local: Mathlib supplies the elementary order and square-root inequalities, while the additional term comes from the project's submeasurement consistency definitions.
- **Key mathematical inputs.** The measurement-valued triangle substitution lemma, the submeasurement total-overlap displacement, and the projective point-consistency transport in the self-improvement final-fields route.

## 2 Key theorem forms for the projective point-consistency transport

This note concerns the projective point-consistency step in the self-improvement theorem of [JNV$^+$20, Section 9].[1] The helper stage produces

[1] Tracked in GitHub issue #1093 and PR #1222. In the TeX source consulted here, the projective point-consistency step is in `references/ldt-paper/self_improvement.tex`, lines 719–726. The substitution proposition used there is `prop:triangle-sub` in `references/ldt-paper/preliminaries.tex`.

a polynomial submeasurement $\widehat{H}$ which is consistent with the point measurement $A$: on average over $u$,

$$A^u_a \otimes I \simeq_{\widehat{\zeta}} I \otimes \widehat{H}_{[h(u)=a]}.$$

After orthonormalization, the paper obtains a projective submeasurement $H$ and a postprocessed state-dependent distance estimate

$$I \otimes \widehat{H}_{[h(u)=a]} \approx_{\widehat{\zeta}_{\text{dataprocess}}} I \otimes H_{[h(u)=a]}.$$

The printed proof then applies `prop:triangle-sub` and concludes

$$A^u_a \otimes I \simeq_{\widehat{\zeta}+\sqrt{\widehat{\zeta}_{\text{dataprocess}}}} I \otimes H_{[h(u)=a]}.$$

# 3 Where the total term enters

In the measurement-valued right-register form of the triangle-substitution argument, the consistency defect for a fixed question has the form

$$\max\left(0,\ \langle\psi, A_{\text{tot}} \otimes B_{\text{tot}}\psi\rangle - \sum_a \langle\psi, A_a \otimes B_a\psi\rangle\right).$$

When $B$ is a measurement, $B_{\text{tot}} = I$. Thus replacing one right-register measurement by another changes only the matching-mass term

$$\sum_a \langle\psi, A_a \otimes B_a\psi\rangle.$$

That term is exactly what the usual Cauchy–Schwarz estimate controls by the square root of the state-dependent distance.

In the projective-output point-consistency step, however, the right-register families are

$$u \longmapsto \widehat{H}_{[h(u)=a]}, \qquad u \longmapsto H_{[h(u)=a]}.$$

They are submeasurements, not measurements. Their total operators are $\widehat{H}_{\text{tot}}$ and $H_{\text{tot}}$, respectively, and these need not both equal the identity. Consequently the replacement also changes the total-overlap term. The additional scalar is

$$\mathbb{E}_u \left|\langle\psi, A^u_{\text{tot}} \otimes H_{\text{tot}}\psi\rangle - \langle\psi, A^u_{\text{tot}} \otimes \widehat{H}_{\text{tot}}\psi\rangle\right|.$$

Since $A^u$ is a measurement, $A^u_{\text{tot}} = I$, but this does not remove the distinction between $H_{\text{tot}}$ and $\widehat{H}_{\text{tot}}$.

## 4 Relation with the formal development

The formal development therefore proves a submeasurement version of the right-register substitution lemma with this total-overlap displacement stated explicitly.[2] In ordinary mathematical terms, the formal lemma says that if the helper point-consistency error is at most $\delta$, the state-dependent distance between the two right-register submeasurement families is at most $\varepsilon$, and the averaged total-overlap displacement is at most $\eta$, then the transported consistency error is at most

$$\delta + \sqrt{\varepsilon} + \eta.$$

This is the same argument as the measurement-valued triangle-substitution proposition, except that the term which vanishes for measurements is retained.

The formal statement does not yet prove an internal bound on $\eta$ from the data-processing estimate. A separate estimate may be possible, for example with a dimension- or outcome-cardinality dependent loss obtained by applying Cauchy–Schwarz to the sum of the outcome differences. Such a bound is not the literal measurement-valued step printed in the paper, and it has not yet been formalized as part of the self-improvement theorem.

## 5 Conclusion

The printed proof applies a measurement-valued substitution principle at a place where the formal objects are submeasurements. The matching-mass part of the argument is unchanged and is controlled by the square root of the state-dependent distance, but a second term involving the right-register totals appears. This term is zero for measurements and is not zero in general for submeasurements.

Thus the current formal development records a genuine proof-level gap in the projective point-consistency paragraph. The corrected formal statement includes an additional total-overlap displacement parameter. Closing the paper step as stated requires either a proof that this displacement is absorbed by the available data-processing error, possibly with a revised numerical

[2] The preliminary statement is triangleSub_right_subMeas_totalGap in `MIPStarRE/LDT/Preliminaries/Triangles/Core.lean`. The self-improvement wrappers are final_fields_point_consistency_totalGap_natural and final_fields_point_consistency_totalGap in `MIPStarRE/LDT/SelfImprovement/Theorems/Results/BoundednessTransport/PointConsistencyLiteral.lean`.

constant, or a different argument which avoids changing the total-overlap term.

## References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.