The Constant in the Completion-Route Orthonormalization Theorem

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1 At a glance

This note concerns the constant in the Section 5 orthonormalization theorem of [JNV+20].

- **Difficulty.** Medium. The discrepancy is a scalar loss, but recovering the paper constant requires aligning the local repair lemma with the bipartite self-consistency hypothesis used in the source proof.
- **Estimated weight.** One reformulated repair theorem, with a moderate amount of project-local scalar and transport work.
- **Mathlib/project split.** Mostly project-local; Mathlib supplies the elementary real-power inequalities and matrix order infrastructure.
- **Key mathematical inputs.** The Section 5 completion step, the conversion from consistency to almost-projectivity, and the local $Q/X/\widehat{X}/P$ repair theorem.

2 Key theorem forms

Throughout, $|\psi\rangle$ denotes a normalized bipartite state on $\mathcal{H}_A \otimes \mathcal{H}_A$, $A = \{A_a\}_{a\in\mathcal{A}}$ is a submeasurement on $\mathcal{H}_A$ with strong self-consistency error $\zeta \geq 0$, and $P = \{P_a\}_{a\in\mathcal{A}}$ is a projective submeasurement on $\mathcal{H}_A$. The relation $\{X_a\} \approx_\delta \{Y_a\}$, in the tensor-placement used here, abbreviates the state-dependent squared-distance estimate

$$\sum_a \langle\psi| \,(X_a \otimes I - Y_a \otimes I)^\dagger (X_a \otimes I - Y_a \otimes I)\, |\psi\rangle \leq \delta.$$

The paper states the orthonormalization theorem with the estimate

$$A_a \otimes I \approx_{100\zeta^{1/4}} P_a \otimes I.$$

The Lean declaration named MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalization now proves this paper statement and its axiom audit requires only the standard Lean axioms. The proved completion-route theorem MIPStarRE.LDT.MakingMeasurementsProjective.orthonorma establishes the same conclusion with the weaker estimate

$$A_a \otimes I \approx_{120\zeta^{1/4}} P_a \otimes I.$$

The earlier input-driven theorem, in which the stronger repair input was supplied separately, has been removed from the public Section 5 API; the corresponding statement is now represented by the paper-facing theorem together with a separate proved completion-route comparison.[1]

3 The source assertion

In the source proof of the orthonormalization lemma, the submeasurement A is first completed to a measurement $\widehat{A}$ on $\mathcal{A} \cup \{\perp\}$, by setting $\widehat{A}_a = A_a$ for $a \in \mathcal{A}$ and $\widehat{A}_\perp = I - \sum_{a\in\mathcal{A}} A_a$. The completed measurement has bipartite strong self-consistency error 2ζ. The paper then applies the main orthonormalization lemma directly at this error level. The relevant source passages are the main lemma statement at `references/ldt-paper/orthonormalization.tex:282-293` (`lem:orthonormalization-main-lemma`) and the completion-and-apply argument at `references/ldt-paper/orthonormalization.tex:304-369`. This gives

$$\widehat{A}_a \otimes I \approx_{84(2\zeta)^{1/4}} \widehat{P}_a \otimes I.$$

The displayed estimate occurs at line 358, and the closing scalar calculation $84 \cdot 2^{1/4} \simeq 99.89 \leq 100$ occurs at line 368. Thus the paper obtains the stated $100\zeta^{1/4}$ bound after discarding the completion outcome.

4 Comparison with the formalization

The proved completion-route theorem follows a slightly different route. After completion it converts the 2ζ self-consistency estimate into a source-almost-projective estimate before invoking the locality-preserving $Q/X/\widehat{X}/P$ repair theorem. This route is recorded by the named Lean declaration MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalizationCompletionRoute

[1] The source theorem is `references/ldt-paper/orthonormalization.tex:67-75` (`thm:orthonormalization`). The relevant declarations are MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalization, MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalizationCompletionRoute, and MIPStarRE.LDT.MakingMeasurementsProjective.leftLiftedProjectivizationRepair_of_sourceAlmostProjective_two_mul. This note is attached to GitHub issue #1032; the constant discrepancy is resolved by proving the source-aligned theorem statement. The heterogeneous `lem:orthonormalization-main-lemma` proof remains a separate tracked obligation, but the public orthonormalization theorem no longer depends on that unfinished lemma. The weaker completion-route construction stays under a separate name.

and the error function $\mathsf{orthonormalizationCompletionRouteError}(\zeta) = 120\zeta^{1/4}$. The conversion doubles the scalar:

$$\text{consistencyToAlmostProjectiveError}(2\zeta) = 4\zeta.$$

The local repair theorem then gives

$$84(4\zeta)^{1/4} = 84 \cdot 4^{1/4}\zeta^{1/4} \leq 120\zeta^{1/4}.$$

Thus the completion-route theorem has the same mathematical shape as the paper theorem, but its constant is weaker by the extra factor $2^{1/4}$ introduced by this conversion step. It is kept as useful proof content, but it is no longer the declaration presented as the paper theorem.

5 Conclusion

The scalar constant discrepancy in the completion-to-measurement route has been removed from the statement of MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalization. The formal proof follows the paper's completion-to-measurement reduction and then uses a sharp locality-preserving repair route at the correct 2ζ scale, avoiding the extra factor introduced by the historical completion-route proof. The weaker completion-route theorem remains available under the separate name MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalizationCompletionRoute.

The heterogeneous MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalizationMainLemma is now also checked with only the standard Lean axioms. Thus the former Section 5 proof-frontier language in this note is historical. The public theorem MIPStarRE.LDT.MakingMeasurementsProjective.orthonormalization has the paper constant, and the weaker completion-route theorem remains only as a separate auxiliary construction.

References

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. Quantum soundness of the classical low individual degree test, 2020. arXiv:2009.12982.