Codes for Simultaneous Transmission of Quantum and Classical Information

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Outline

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- **Advantage** compared to independent solutions?

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- **Advantage** compared to independent solutions?
- For the finite length case: in Kremsky et al. [2008], the authors consider the problem in the context of so-called entanglement-assisted codes. The examples given in Kremsky et al. [2008], e.g. $[[9, 1 : 2, 3]]$, however, fail to demonstrate an advantage compared to stabilizer quantum codes. (Even $[[8, 3, 3]]$ exists)

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- A **general construction** $\Rightarrow$ up to 34 qubits. (See arXiv version: 1701:06963)
- **Linear program bound** on n, k, m, d

Background and Notations

Our discussion is based on the theory of stabilizer quantum codes and its connection to classical error-correcting codes (see, e.g., Calderbank et al. [1998]). We use the following notations.

- $((n, K, d))_q$
- $[\![n, k, d]\!]_q$
- $(n, M, d)_q$
- $[n, m, d]_q$
- $[\![n, k{:}m, d]\!]_q$
- $((n, K{:}M, d))_q$

Trivial Construction

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- $[\![n_1, k_1, d]\!]_q + [n_2, m_2, d]_q \Rightarrow [\![n_1+n_2, k_1{:}m_2, d]\!]_q$

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Our goal is to find codes that have better parameters than the codes that can be obtained by these trivial constructions.

Error Correction Conditions

A hybrid quantum code $\mathcal{C} = ((n, K{:}M))_q$ can be described by a collection

$$\{\mathcal{C}^{(\nu)} \colon \nu = 1, \ldots, M\}$$

of M quantum codes $\mathcal{C}^{(\nu)} = ((n, K, d))_q$. The classical information ν determines which quantum code $\mathcal{C}^{(\nu)}$ is used to encode the quantum information.
In the following, we will use Greek letters when referring to classical information. Assume that $\{|c_i^{(\nu)}\rangle \colon i = 1, \ldots, K\}$ is an orthonormal basis for the code $\mathcal{C}^{(\nu)}$.

Error Correction Conditions

In order to be able to correct the linear span of error operators $\{E_k \colon k = 1, 2, \ldots\}$, each of the codes $\mathcal{C}^{(\nu)}$ has to obey the Knill-Laflamme conditions Knill and Laflamme [1997], i. e.,

$$\langle c_i^{(\nu)} | E_k^\dagger E_\ell | c_j^{(\nu)} \rangle = \alpha_{k\ell}^{(\nu)} \delta_{ij}.$$

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Note that the constants $\alpha_{k\ell}^{(\nu)} \in C$ may depend on the classical information ν. To retrieve the classical information ν, one has to be able to perfectly distinguish the states $|c_i^{(\nu)}\rangle$ and $|c_j^{(\mu)}\rangle$ for $\nu \neq \mu$ and arbitrary i and j after an error.

$$\langle c_i^{(\nu)} | E_k^\dagger E_\ell | c_j^{(\mu)} \rangle = 0, \text{ for } \mu \neq \nu.$$

Error Correction Conditions

Theorem

A hybrid quantum code $\mathcal{C} = ((n, K{:}M))_q$ with orthonormal basis states $\{|c_i^{(\nu)}\rangle : i = 1, \ldots, K,\ \nu = 1, \ldots, M\}$ can correct all errors $\{E_k : k = 1, 2, \ldots\}$ if and only if

$$\langle c_i^{(\nu)} | E_k^\dagger E_\ell | c_j^{(\mu)} \rangle = \alpha_{k\ell}^{(\nu)} \delta_{ij} \delta_{\mu\nu}.$$

Discussions:

- When $\alpha_{k\ell}^{(\nu)}$ do not depend on ν, condition reduces to Knill-Laflamme condition for a quantum code $\mathcal{C} = ((n, KM))_q$.

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- For hybrid codes with better parameters, there should be at least a pair ν, μ and errors E_k, E_ℓ such that $\alpha_{k\ell}^{(\nu)} \neq \alpha_{k\ell}^{(\mu)}$.

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- When $\alpha_{k\ell}^{(\nu)}$ do not depend on ν, condition reduces to Knill-Laflamme condition for a quantum code $\mathcal{C} = ((n, KM))_q$.
- For hybrid codes with better parameters, there should be at least a pair ν, μ and errors E_k, E_ℓ such that $\alpha_{k\ell}^{(\nu)} \neq \alpha_{k\ell}^{(\mu)}$.
- When the error operators E_k are unitary, $\alpha_{kk}^{(\nu)} = 1$. Then $\alpha_{k\ell}^{(\nu)} \neq 0$ for some ν and $k \neq \ell$, which suggests that some of the codes $\mathcal{C}^{(\nu)}$ might be taken to be **degenerate codes.**

Code Construction

We outline the construction of hybrid quantum codes in the framework of CWS codes/union stabilizer codes. We start with a quantum code $\mathcal{C}^{(0)} = ((n, K, d))_q$ which is a CWS code that might even be a stabilizer code $\mathcal{C}^{(0)} = [\![n, k, d]\!]_q$.

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$$\mathcal{C}^{(\nu)} = t_\nu \mathcal{C}^{(0)}$$

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(a) $S \Rightarrow$ self orthogonal classical code C_0.

(b) $C_0 \subseteq C_0^* \Rightarrow N$

$$d = \min\{\mathrm{wgt}\, c \colon c \in C_0^* \setminus C_0\} > \min\{\mathrm{wgt}\, c \colon c \in C_0^* \setminus \{0\}\}.$$

The codes $\mathcal{C}^{(\nu)} = t_\nu \mathcal{C}^{(0)}$ are associated with cosets $C_0^* + t_\nu$ of the normalizer code C_0^*,

Code Construction

When the cosets $C_0^* + t_\nu$ and $C_0^* + t_\mu$ are different, then the codes $\mathcal{C}^{(\nu)}$ and $\mathcal{C}^{(\mu)}$ will be orthogonal to each other. The hybrid code $\mathcal{C}$ is associated with the classical code

$$C^* = \bigcup_{\nu=1}^{M} C_0^* + t_\nu.$$

When the union of the codes is an additive code, the hybrid quantum code will be a stabilizer code.

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It turns out that the minimum distance of a hybrid code associated with the codes $C_0 \le C^*$ is given by

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Note that the minimum(d) is taken over a smaller set compared to d', as $C \leq C_0$, and hence $d \geq d'$.

Code Construction

In summary, we have the following construction.

Theorem

Let $C_0 = (n, q^{n-k}, d_0)_{q^2}$ be a classical additive code that is contained in its symplectic dual C_0^*. Further, let $C^* = (n, q^{n+k+m}, d')_{q^2}$ be an additive code containing C_0^*. Then there exists a hybrid stabilizer code $\mathcal{C} = [\![n, k{:}m, d]\!]_q$ encoding k qudits and m classical symbols. The minimum distance of $\mathcal{C}$ is given by

$$d = \min\{\mathrm{wgt}\, c \colon c \in C^* \setminus C_0\}.$$

LP Bound(Method)

In order to obtain bounds on the parameters of hybrid stabilizer codes $[\![n, k{:}m, d]\!]_q$, we consider the homogeneous weight enumerators of the associated code C_0 and its symplectic dual C_0^*, as well as the code C^* and its symplectic dual C:

$$\mathcal{W}_{C_0}(X,Y) = \sum_{w=0}^{n} A_w^{\perp} X^{n-w} Y^w, \; \mathcal{W}_{C_0^*}(X,Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w,$$

$$\mathcal{W}_{C}(X,Y) = \sum_{w=0}^{n} B_w^{\perp} X^{n-w} Y^w, \mathcal{W}_{C^*}(X,Y) = \sum_{w=0}^{n} B_w X^{n-w} Y^w.$$

LP Bound(Method)

The weight enumerators of C_0 and C_0^*, as well as those of C and C^*, are related by the MacWilliams transformation, i. e.,

$$\mathcal{W}_{C_0^*}(X,Y) = \frac{1}{|C_0|}\mathcal{W}_{C_0}\left(X + (q^2-1)Y, X - Y\right),$$
$$\mathcal{W}_{C^*}(X,Y) = \frac{1}{|C|}\mathcal{W}_{C}\left(X + (q^2-1)Y, X - Y\right).$$

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More details can be found in the proceedings, including tables.

Results (Code Search)

Search for $\mathcal{C} = [\![n, k{:}m, d]\!]_2$ codes with distance $d \geq 3$.

- Union Stabilizer:
  1. Start with the self-dual codes from the classification in Danielsen, Danielsen and Parker [2006].
  2. Construct impure quantum codes $[\![n, 1, d]\!]_2$ Then look for additional vectors for the encoding of classical information, resulting in an $[\![n, 1{:}m', d]\!]_2$ hybrid code.
  3. In some cases, the code $[\![n, 1{:}m', d]\!]_2$ is in fact a $[\![n, k{:}m' - k + 1, d]\!]_2$.

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  3. In some cases, the code $[\![n, 1{:}m', d]\!]_2$ is in fact a $[\![n, k{:}m' - k + 1, d]\!]_2$.
- CWS Framework:
  - **(a)** start with the graph state from the classification in Danielsen, Danielsen and Parker [2006].
  - **(b)** Construct impure code using CWS framework, then look for additional vectots for the encoding of classical information by searching for MAX-Clique. Results in a hybrid code with parameters $[\![n, k{:}m'', d]\!]_2$
  - **(c)** $\Pi_i E_k^\dagger E_l \Pi_j = 0, i \neq j$

Results

Theorem

There exist hybrid codes with the following parameters:

$$[\![7,1{:}1,3]\!]_2,\quad [\![9,2{:}2,3]\!]_2,\quad [\![10,3{:}2,3]\!]_2, [\![11,4{:}2,3]\!]_2,$$
$$[\![11,1{:}2,4]\!]_2, [\![13,1{:}4,4]\!]_2,$$
$$[\![13,1{:}1,5]\!]_2, [\![14,1{:}2,5]\!]_2, [\![15,1{:}3,5]\!]_2,$$
$$[\![19,9{:}1,4]\!]_2, [\![20,9{:}2,4]\!]_2, [\![21,9{:}3,4]\!]_2, [\![22,9{:}4,4]\!]_2 \cdots$$

All these codes have better parameters than codes obtained from the best quantum codes using trivial construction.

Results (Seven qubits)

$$\left(\begin{array}{ccccccc}
X & I & I & Z & Y & Y & Z \\
Z & I & I & I & I & I & X \\
I & X & I & X & Z & I & I \\
I & Z & I & Z & I & X & X \\
I & I & X & X & I & Z & I \\
I & I & Z & Z & X & I & X \\
\hline
I & I & I & X & Z & Z & X \\
I & I & I & Z & X & X & I \\
\hline\hline
I & I & I & I & X & Y & Y
\end{array}\right)$$

- No $[\![7,2,3]\!]_2$
- Starting with this impure code, we obtain a hybrid code with parameters $[\![7,1{:}1,3]\!]_2$.
- The additional generator that is used to encode one classical bit is given below the double horizontal line.
- We have not found a $[\![7,1{:}2,3]\!]_2$ which is not ruled out by linear programming.

Results (Eight qubits)

- For eight qubits, there is a quantum code with parameters $[\![8, 3, 3]\!]_2$. Using trivial construction, we obtain an optimal hybrid code with parameters $[\![8, 2{:}1, 3]\!]_2$, as well as a code $[\![8, 1{:}2, 3]\!]_2$.
- We have not found a hybrid code with parameters $[\![8, 1{:}3, 3]\!]_2$ that might exist.

Results (Nine qubits)

$$\begin{pmatrix}
X & I & I & Z & Y & Z & X & X & Y \\
Z & I & I & I & I & X & I & I & I \\
I & X & I & Z & Y & I & Y & I & Z \\
I & Z & I & I & I & I & X & I & I \\
I & I & X & Z & Z & I & I & I & X \\
I & I & Z & I & Y & X & I & Y & I \\
I & I & I & X & X & X & I & Z & I \\
\hline
I & I & I & Z & I & I & X & Y & X \\
I & I & I & I & X & I & I & Z & Y \\
I & I & I & I & Z & I & I & X & X \\
I & I & I & I & I & X & X & I & X \\
\hline\hline
I & I & I & I & I & Z & I & Z & X \\
I & I & I & I & I & I & Y & X & Z
\end{pmatrix}$$

- For nine qubits, we found a hybrid code $[\![9, 2{:}2, 3]\!]_2$
- Taking all possible products of the two generators below the double horizontal line we obtain the four translation operators $t^{(1)} = id$, $t^{(2)}$, $t^{(3)}$, and $t^{(4)} = t^{(2)}t^{(3)}$ used to encode two extra classical bits.

Results (10 qubits)

- A hybrid code $[\![10, 3{:}2, 3]\!]_2$ exists.
- Via linear programming it is found that this code is optimal in the sense that it encodes the maximal possible number m of additional classical bits among all codes $[\![10, 3{:}m, 3]\!]_2$.

Results (11 qubits)

The first non-trivial hybrid code with distance $d = 4$ has been found for eleven qubits. A hybrid code $[\![11, 1{:}2, 4]\!]_2$ is given. We found a hybrid code $[\![11, 4{:}2, 3]\!]_2$ as well.

Results (Appending construction)

$$
\left(\begin{array}{ccccccccccc:cc}
\multicolumn{11}{c:}{\text{the stabilizer part of } [\![11, 1{:}2, 4]\!]_2} & I & I \\
\hdashline
I & I & I & I & I & I & I & I & I & I & I & Z & I \\
I & I & I & I & I & I & I & I & I & I & I & I & Z \\
\hline
I & I & I & I & I & Z & I & X & I & X & X & I & I \\
I & I & I & I & I & I & Z & Z & X & X & I & I & I \\
\hline\hline
I & X & I & I & I & I & I & I & X & Y & X & X & X \\
I & I & I & I & I & I & X & I & X & I & I & X & X \\
I & I & I & I & I & I & I & X & Y & X & Y & X & X \\
I & I & I & I & I & I & I & X & I & Y & Y & X & I
\end{array}\right)
$$

Appending two qubits in the state $|0\rangle$ to the impure quantum code $[\![11, 1, 4]\!]_2$ given above the double horizontal line, one obtains an impure code $[\![13, 1, 4]\!]_2$. This code can additionally transmit four classical bits, i. ,e., one obtains the hybrid code $[\![13, 1{:}4, 4]\!]_2$.

Results (Appending construction)

We generalize this construction by following theorem.

Theorem

Let $\mathcal{C}_1 = [\![n, k_1, d_1]\!]_q \subset \mathcal{C}_2 = [\![n, k_2, d_2]\!]_q$ be nested quantum codes. Further, let $C_3 = [n_3, k_2 - k_1, d_3]_q$ be a classical linear code. Then there is a hybrid quantum code $\mathcal{C} = [\![n + n_3, k_1{:}(k_2 - k_1), d]\!]_q$ with $d \geq \min(d_1, d_2 + d_3)$.

From the nested stabilizer codes $[\![11, 1, 5]\!]_2 \subset [\![11, 4, 3]\!]_2$ and classical codes $[n_3, n_3 - 1, 2]_2$, one obtains hybrid codes $[\![13, 1{:}1, 5]\!]_2$, $[\![14, 1{:}2, 5]\!]_2$, and $[\![15, 1{:}3, 5]\!]_2$. Similarly, from $[\![17, 9, 4]\!]_2 \subset [\![17, 13, 2]\!]_2$, one gets $[\![19, 9{:}1, 4]\!]_2$, $[\![20, 9{:}2, 4]\!]_2$, $[\![21, 9{:}3, 4]\!]_2$, and $[\![22, 9{:}4, 4]\!]_2$.

Discussion

- The code conditions derived here suggest that one should start with good impure quantum codes.
- In order to find a direct construction of hybrid codes with good parameters, a first step could be to develop methods to construct **good non-trivial impure codes**
- How?

Conclusions

- We consider the characterization as well as the construction of quantum codes that allow to transmit both quantum and classical information, which we refer to as "**hybrid codes**".

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- We construct hybrid codes $[\![n, k{:}m, d]\!]_q$ with length n and distance d, that simultaneously transmit k qudits and m symbols from a classical alphabet of size q.

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- We consider the characterization as well as the construction of quantum codes that allow to transmit both quantum and classical information, which we refer to as "**hybrid codes**".
- We construct hybrid codes $[\![n, k{:}m, d]\!]_q$ with length n and distance d, that simultaneously transmit k qudits and m symbols from a classical alphabet of size q.
- Many good codes up to 34 qubits have been found. All these codes have better parameters than hybrid codes obtained from the best known stabilizer quantum codes.

Conclusions

- We consider the characterization as well as the construction of quantum codes that allow to transmit both quantum and classical information, which we refer to as "**hybrid codes**".
- We construct hybrid codes $[\![n, k{:}m, d]\!]_q$ with length n and distance d, that simultaneously transmit k qudits and m symbols from a classical alphabet of size q.
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Thank you!

Questions/Answers

LP Bound(Method)

Additionally, we have:

$$A_0^{\perp} = A_0 = B_0 = 1,$$

$$\sum_{w=0}^{n} A_w^{\perp} = q^{n-k}, \qquad \sum_{w=0}^{n} A_w = q^{n+k},$$

$$\sum_{w=0}^{n} B_w^{\perp} = q^{n-k-m}, \qquad \sum_{w=0}^{n} B_w = q^{n+k+m}.$$

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When a hybrid stabilizer code $[\![n, k{:}m, d]\!]_q$ exists, the linear program for the variables $B_w^{\perp}$, $A_w^{\perp}$, A_w, and B_w has an integer solution.

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When a hybrid stabilizer code $[\![n, k{:}m, d]\!]_q$ exists, the linear program for the variables $B_w^\perp$, $A_w^\perp$, A_w, and B_w has an integer solution. For qubit codes, we can strengthen the LP by additionally considering the shadow enumerator Rains [1999]

$$\mathcal{S}_{C_0}(X, Y) = \frac{1}{|C_0|}\mathcal{W}_{C_0}\left(X + (q^2 - 1)Y, Y - X\right),$$

which has to have non-negative integer coefficients. Ref to

LP Bound

Using CPLEX V12.6.3.0, we checked whether the integer program is feasible. More precisely,

- we first fix the length n, number of qudits k, and number $M = 2^m$ of classical symbols.
- Then we look for the largest minimum distance d for which the integer program is found to be feasible.
- The resulting bounds on the parameters $[\![n, k{:}m, d]\!]_2$ are listed in Table, i. ,e., for fixed parameters n, k, and d, the largest possible value for m is given.
- For $n > 14$, there seem to be some precision issues, so we list only the bounds for $n \leq 14$.

LP Bound($d = 3$)

n \ k	0	1	2	3	4	5	6	7	8
5	2	0	–	–	–	–	–	–	–
6	3	0	–	–	–	–	–	–	–
7	4	2	–	–	–	–	–	–	–
8	4	3	1	0	–	–	–	–	–
9	5	4	3	1	–	–	–	–	–
10	6	5	4	2	1	–	–	–	–
11	7	6	5	4	2	0	–	–	–
12	8	7	6	5	3	2	0	–	–
13	9	8	7	5	5	3	1	0	–
14	10	9	8	7	6	5	3	1	0

LP Bound($d = 4$)

n \ k	0	1	2	3	4	5	6
5	1	–	–	–	–	–	–
6	2	–	–	–	–	–	–
7	3	–	–	–	–	–	–
8	4	–	–	–	–	–	–
9	4	–	–	–	–	–	–
10	5	3	1	–	–	–	–
11	6	4	2	–	–	–	–
12	7	5	4	2	0	–	–
13	8	6	5	4	2	0^*	–
14	9	6	6	5	3	2	0

LP Bound($d = 5$)

n \ k	0	1	2	3
5	1	–	–	–
6	1	–	–	–
7	1	–	–	–
8	2	–	–	–
9	2	–	–	–
10	3	–	–	–
11	4	0	–	–
12	4	2	–	–
13	5	4	–	–
14	6	5	3	1